

The C^* -algebra of $SL(2, \mathbb{R})$

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Abstract

The C^* -algebra of the group $SL(2, \mathbb{R})$ is characterized using the operator valued Fourier transform. In particular, it is shown by explicit computations, that the Fourier transform of this C^* -algebra fulfills the norm controlled dual limit property.

1 Introduction

In this article, the structure of the C^* -algebra of the group $G = SL(2, \mathbb{R})$ will be analyzed. The structure of the group C^* -algebras is already known for certain classes of Lie groups: The C^* -algebras of the Heisenberg and the thread-like Lie groups have been analyzed in [8] and the C^* -algebras of the $ax + b$ -like groups in [7]. Furthermore, the C^* -algebras of the 5-dimensional nilpotent Lie groups have been determined in [10] and H.Regeiba characterized the C^* -algebras of all 6-dimensional nilpotent Lie groups in his doctoral thesis (see [9]). Just recently, the C^* -algebras of connected real two-step nilpotent Lie groups have been analyzed in [4].

For semisimple Lie groups, there is no explicit description of their group C^* -algebras given in literature. However, for those semisimple Lie groups whose unitary dual is classified, the procedure of the determination of the group C^* -algebra used in this article might be successfully applied in a similar way. A characterization of reduced group C^* -algebras of semisimple Lie groups can be found in [13].

In the present paper, the group C^* -algebra of $SL(2, \mathbb{R})$ shall be described very explicitly. It will be shown that it is characterized by some conditions which are called "norm controlled dual limit conditions" and which will be given in Section 3 below. In an abstract existence result in [1] these conditions are shown to hold true for all simply connected connected nilpotent Lie groups. They are explicitly checked for all 5- and 6-dimensional nilpotent Lie groups (see [10]), for the Heisenberg Lie groups and the thread-like Lie groups (see [8]) and for the connected real two-step nilpotent Lie groups (see [4]).

At the beginning of this article, some notations and important facts which are needed in order to determine the C^* -algebra of $SL(2, \mathbb{R})$ will be recalled. In Section 3 the above mentioned conditions which are characterizing a group C^* -algebra will be defined. The main result of this article, namely the compliance of the group C^* -algebra of $SL(2, \mathbb{R})$ with these conditions, will be formulated and its proof will be accomplished in the following sections. Section 4 is about the unitary dual of $SL(2, \mathbb{R})$ and its topology and in Section 5 and 6, the above specified

conditions will be verified for the group $G = SL(2, \mathbb{R})$. Finally, in Section 7, an alternative version of the result about the C^* -algebra of $SL(2, \mathbb{R})$ will be presented and the concrete structure of $C^*(G)$ will be given.

2 Preliminaries

2.1 General definitions

Definition 2.1 (Fourier transform).

The Fourier transform $\mathcal{F}(a) = \hat{a}$ of an element a of a C^* -algebra C is defined in the following way: One chooses for every $\gamma \in \widehat{C}$, the unitary dual of C , a representation $(\pi_\gamma, \mathcal{H}_\gamma)$ in the equivalence class of γ and defines

$$\mathcal{F}(a)(\gamma) := \pi_\gamma(a) \in \mathcal{B}(\mathcal{H}_\gamma).$$

Then $\mathcal{F}(a)$ is contained in the algebra of all bounded operator fields over \widehat{C}

$$l^\infty(\widehat{C}) = \left\{ \phi = (\phi(\pi_\gamma) \in \mathcal{B}(\mathcal{H}_\gamma))_{\gamma \in \widehat{C}} \mid \|\phi\|_\infty := \sup_{\gamma \in \widehat{C}} \|\phi(\pi_\gamma)\|_{op} < \infty \right\}$$

and the mapping

$$\mathcal{F} : C \rightarrow l^\infty(\widehat{C}), \quad a \mapsto \hat{a}$$

is an isometric $*$ -homomorphism.

Definition 2.2 (Properly converging sequence).

A sequence $(t_k)_{k \in \mathbb{N}}$ in a topological space is called properly converging, if $(t_k)_{k \in \mathbb{N}}$ has limit points and if every subsequence of $(t_k)_{k \in \mathbb{N}}$ has the same limit set as $(t_k)_{k \in \mathbb{N}}$.

Recall that the C^* -algebra of a locally compact group G is defined as the completion of the convolution algebra $L^1(G)$ with respect to the C^* -norm of $L^1(G)$, i.e.

$$C^*(G) := \overline{L^1(G)}^{\|\cdot\|_{C^*(G)}} \quad \text{with} \quad \|f\|_{C^*(G)} := \sup_{\pi \in \widehat{G}} \|\pi(f)\|_{op}.$$

A well-known result, that can be found in [3], states that the unitary dual of $C^*(G)$ coincides with the unitary dual of G :

$$\widehat{C^*(G)} = \widehat{G}.$$

The unitary dual \widehat{G} has a natural topology which can be characterized in the following way:

Theorem 2.3 (Topology of the dual space).

Let $(\pi_k, \mathcal{H}_{\pi_k})_{k \in \mathbb{N}}$ be a family of irreducible unitary representations of a locally compact group G . Then $(\pi_k)_{k \in \mathbb{N}}$ converges to (π, \mathcal{H}_π) in \widehat{G} if and only if for some non-zero (respectively for every) vector ξ in \mathcal{H}_π , for every $k \in \mathbb{N}$ there exists $\xi_k \in \mathcal{H}_{\pi_k}$ such that the sequence of matrix coefficients $(\langle \pi_k(\cdot)\xi_k, \xi_k \rangle)_{k \in \mathbb{N}}$ converges uniformly on compacta to the matrix coefficient $\langle \pi(\cdot)\xi, \xi \rangle$.

The proof of this theorem can be found in [3].

2.2 The Lie group $SL(2, \mathbb{R})$

From now on let

$$G := SL(2, \mathbb{R}) = \{A \in M(2, \mathbb{R}) \mid \det A = 1\}$$

and let

$$K := SO(2) = \left\{ k_\varphi := \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \mid \varphi \in [0, 2\pi) \right\}$$

be its maximal compact subgroup. Furthermore, define the one-dimensional subgroups

$$N := \left\{ \mu_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\} \quad \text{and} \quad A := \left\{ a_t := \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

of G . Let

$$\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) = \{A \in M(2, \mathbb{R}) \mid \operatorname{tr} A = 0\}$$

be the Lie algebra of G .

From the Iwasawa decomposition, $G = KAN$ and thus, for every $g \in G$ there exist $\kappa(g) \in K$, $\mu \in N$ and $H(g) \in \mathfrak{a}$, where $\mathfrak{a} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mid t \in \mathbb{R} \right\}$ is the Lie algebra of A , such that

$$g = \kappa(g)e^{H(g)}\mu.$$

Moreover, define on \mathfrak{a} the mappings ρ and ν_z for $z \in \mathbb{C}$ as

$$\rho \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} := t \quad \text{and} \quad \nu_z \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} := zt \quad \forall t \in \mathbb{R}.$$

Furthermore, let

$$L^2(K)_+ := \{f \in L^2(K, \mathbb{C}) \mid f(k) = f(-k) \forall k \in K\} \quad \text{and}$$

$$L^2(K)_- := \{f \in L^2(K, \mathbb{C}) \mid f(k) = -f(-k) \forall k \in K\}$$

and define for every $u \in \mathbb{C}$ the representations $\mathcal{P}^{+,u}$ on $\mathcal{H}_{\mathcal{P}^{+,u}} := L^2(K)_+$ and $\mathcal{P}^{-,u}$ on $\mathcal{H}_{\mathcal{P}^{-,u}} := L^2(K)_-$ as

$$\mathcal{P}^{\pm,u}(g)f(k) := e^{-(\nu_u + \rho)H(g^{-1}k)}f(\kappa(g^{-1}k)) \quad \forall g \in G \quad \forall f \in L^2(K)_\pm \quad \forall k \in K.$$

Remark 2.4.

The representation $(\mathcal{P}^{+,u}, \mathcal{H}_{\mathcal{P}^{+,u}})$ is irreducible if and only if $u \notin 2\mathbb{Z}+1$ and the representation $(\mathcal{P}^{-,u}, \mathcal{H}_{\mathcal{P}^{-,u}})$ is irreducible if and only if $u \notin 2\mathbb{Z}$.

Furthermore, $(\mathcal{P}^{+,u}, \mathcal{H}_{\mathcal{P}^{+,u}})$ and $(\mathcal{P}^{-,u}, \mathcal{H}_{\mathcal{P}^{-,u}})$ are unitary for $u \in i\mathbb{R}$.

For the proof see [5], Chapter 2.

Convention 2.5.

Throughout this paper, by $L^2(K)$ and $C^\infty(K)$ is meant $L^2(K, \mathbb{C})$ and $C^\infty(K, \mathbb{C})$, respectively.

Lemma 2.6.

For every function $f \in L^2(K)_\pm$ and every $g \in G$

$$\int_K e^{-2\rho H(g^{-1}k)} \left| f\left(\kappa(g^{-1}k)\right) \right|^2 dk = \|f\|_{L^2(K)}^2.$$

The proof can be found in [5], Chapter VII.2.

Definition 2.7 (n -th isotypic component).

For a representation $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}})$ of K define for every $n \in \mathbb{Z}$ the n -th isotypic component or K -type of $\tilde{\pi}$ as

$$\mathcal{H}_{\tilde{\pi}}(n) := \{v \in \mathcal{H}_{\tilde{\pi}} \mid \tilde{\pi}(k_\varphi)v = e^{in\varphi}v \ \forall \varphi \in [0, 2\pi)\}.$$

A representation $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}})$ of G is called even (respectively odd), if $\mathcal{H}_{\tilde{\pi}|_K}(n) = \{0\}$ for all odd n (respectively for all even n).

Every irreducible unitary representation of G is even or odd.

Furthermore, the algebraic direct sum

$$\bigoplus_{n \in \mathbb{Z}} \mathcal{H}_{\tilde{\pi}}(n)$$

is dense in $\mathcal{H}_{\tilde{\pi}}$.

Remark 2.8.

By the definition of the Hilbert spaces $L^2(K)_\pm$ of $\mathcal{P}^{\pm,u}$ for $u \in \mathbb{C}$, it is easy to verify that $\mathcal{P}^{+,u}$ is even for every $u \in \mathbb{C}$ and that $\mathcal{P}^{-,u}$ is odd for every $u \in \mathbb{C}$.

By the definition of the n -th isotypic component, one can remark that

$$\mathcal{H}_{\mathcal{P}^{+,u}}(n) = \mathbb{C} \cdot e^{-in\cdot} \text{ for all even } n \in \mathbb{Z} \quad \text{and} \quad \mathcal{H}_{\mathcal{P}^{-,u}}(n) = \mathbb{C} \cdot e^{-in\cdot} \text{ for all odd } n \in \mathbb{Z}.$$

Definition 2.9 (p_n).

Denote for $n \in \mathbb{Z}$ by $b_n(f)$ the n -th Fourier coefficient of $f \in L^2(K)_\pm$ which is defined as

$$b_n(f) := \frac{1}{|K|} \int_K f(k_\varphi) e^{-in\varphi} dk_\varphi,$$

and let

$$p_n(f) := b_{-n}(f) e^{-in\cdot}.$$

One can easily show that for every $u \in \mathbb{C}$ and for every $n \in \mathbb{Z}$ the operator p_n is the projection from $\mathcal{H}_{\mathcal{P}^{\pm,u}} = L^2(K)_\pm$ to the n -th isotypic component of the representation $\mathcal{P}^{\pm,u}$.

3 C^* -algebras with norm controlled dual limits

In this section, the definition of a C^* -algebra with "norm controlled dual limits", which was mentioned in the introduction, will be given.

Definition 3.1.

A C^* -algebra C is called C^* -algebra with **norm controlled dual limits** if it fulfills the following conditions:

- **Condition 1:** Stratification of the unitary dual:

- (a) There is a finite increasing family $S_0 \subset S_1 \subset \dots \subset S_r = \widehat{C}$ of closed subsets of the unitary dual \widehat{C} of C in such a way that for $i \in \{1, \dots, r\}$ the subsets $\Gamma_0 := S_0$ and $\Gamma_i := S_i \setminus S_{i-1}$ are Hausdorff in their relative topologies and such that S_0 consists of all the characters of C .
- (b) For every $i \in \{0, \dots, r\}$ there is a Hilbert space \mathcal{H}_i and for every $\gamma \in \Gamma_i$ there is a concrete realization $(\pi_\gamma, \mathcal{H}_i)$ of γ on the Hilbert space \mathcal{H}_i .

- **Condition 2:** CCR C^* -algebra:

C is a separable CCR (or liminal) C^* -algebra, i.e. a separable C^* -algebra such that the image of every irreducible representation (π, \mathcal{H}) of C is contained in the algebra of compact operators $\mathcal{K}(\mathcal{H})$ (which implies that the image equals $\mathcal{K}(\mathcal{H})$).

- **Condition 3:** Changing of layers:

Let $a \in C$.

- (a) The mappings $\gamma \mapsto \mathcal{F}(a)(\gamma)$ are norm continuous on the different sets Γ_i .
- (b) For any $i \in \{0, \dots, r\}$ and for any converging sequence contained in Γ_i with limit set outside Γ_i (thus in S_{i-1}), there is a properly converging subsequence $\overline{\gamma} = (\gamma_j)_{j \in \mathbb{N}}$, as well as a constant $c > 0$ and for every $j \in \mathbb{N}$ an involutive linear mapping $\tilde{\nu}_j = \tilde{\nu}_{\overline{\gamma}, j} : CB(S_{i-1}) \rightarrow \mathcal{B}(\mathcal{H}_i)$, which is bounded by $c \|\cdot\|_{S_{i-1}}$ (uniformly in j), such that

$$\lim_{j \rightarrow \infty} \|\mathcal{F}(a)(\gamma_j) - \tilde{\nu}_j(\mathcal{F}(a)|_{S_{i-1}})\|_{op} = 0.$$

Here $CB(S_{i-1})$ is the $*$ -algebra of all the uniformly bounded fields of operators $(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_i))_{\gamma \in \Gamma_l, l=0, \dots, i-1}$, which are operator norm continuous on the subsets Γ_l for every $l \in \{0, \dots, i-1\}$, provided with the infinity-norm

$$\|\psi\|_{S_{i-1}} := \sup_{\gamma \in S_{i-1}} \|\psi(\gamma)\|_{op}.$$

Theorem 3.2.

The C^* -algebra of $G = SL(2, \mathbb{R})$ has norm controlled dual limits.

Remark 3.3.

Throughout the rest of this article, this theorem will be proved. Concrete subsets Γ_i and S_i of $\widehat{C^*(G)} = \widehat{G}$ will be defined and in Section 6, the mappings $(\tilde{\nu}_j)_{j \in \mathbb{N}}$ will be constructed.

The norm controlled dual limit conditions completely characterize the structure of a group C^* -algebra in the following sense: Taking the number r , the Hilbert spaces \mathcal{H}_i , the sets Γ_i and S_i for $i \in \{0, \dots, r\}$ (see Section 4.5 and Section 4.2 for their construction) and the mappings $(\tilde{\nu}_j)_{j \in \mathbb{N}}$ (see Section 6 for their construction) required in the above definition, by [10], Theorem 3.5, one gets the result below for the C^* -algebra of $G = SL(2, \mathbb{R})$:

Theorem 3.4.

The C^* -algebra $C^*(G)$ of $G = SL(2, \mathbb{R})$ is isomorphic (under the Fourier transform) to the set of all operator fields φ defined over \widehat{G} such that

1. $\varphi(\gamma) \in \mathcal{K}(\mathcal{H}_i)$ for every $i \in \{1, \dots, r\}$ and every $\gamma \in \Gamma_i$.
2. $\varphi \in l^\infty(\widehat{G})$.
3. The mappings $\gamma \mapsto \varphi(\gamma)$ are norm continuous on the different sets Γ_i .
4. For any sequence $(\gamma_j)_{j \in \mathbb{N}} \subset \widehat{G}$ going to infinity $\lim_{j \rightarrow \infty} \|\varphi(\gamma_j)\|_{\text{op}} = 0$.
5. For every $i \in \{1, \dots, r\}$ and any properly converging sequence $\overline{\gamma} = (\gamma_j)_{j \in \mathbb{N}} \subset \Gamma_i$ whose limit set is contained in S_{i-1} (taking a subsequence if necessary) and for the mappings $\tilde{\nu}_j = \tilde{\nu}_{\overline{\gamma}, j} : CB(S_{i-1}) \rightarrow \mathcal{B}(\mathcal{H}_i)$, one has

$$\lim_{j \rightarrow \infty} \|\varphi(\gamma_j) - \tilde{\nu}_j(\varphi|_{S_{i-1}})\|_{\text{op}} = 0.$$

At the end of this article, an equivalent, but much simpler description of $C^*(G)$ will be given in Theorem 7.1.

4 The unitary dual of $SL(2, \mathbb{R})$

4.1 Introduction of the operator K_u

Now, an operator K_u which is needed in order to describe the unitary dual of G will be introduced using the Knapp-Stein operator:

Define

$$\begin{aligned} C^\infty(K)_+ &:= \{f \in C^\infty(K) \mid f(k) = f(-k) \ \forall k \in K\} \quad \text{and} \\ C^\infty(K)_- &:= \{f \in C^\infty(K) \mid f(k) = -f(-k) \ \forall k \in K\} \end{aligned}$$

and let

$$J_u : C^\infty(K)_+ \rightarrow C^\infty(K)_+ \quad \text{for } u \in \mathbb{C} \text{ with } \operatorname{Re} u > 0$$

be the Knapp-Stein intertwining operator, which sends the representation $\mathcal{P}^{+,u}$ to the representation $\mathcal{P}^{+,-u}$. Furthermore, let $w := k_{\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and extend $f \in L^2(K)$ to G by using the Iwasawa decomposition $G \ni g = \kappa(g)e^{H(g)}\mu$ for $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ and $\mu \in N$. Then, defining $\tilde{f}_u(\kappa(g)e^{H(g)}\mu) := e^{-(\nu_u + \rho)H(g)} f(\kappa(g))$, this operator can be written as

$$J_u f(k) = \int_N \tilde{f}_u(k\mu w) d\mu \quad \forall f \in C^\infty(K)_+ \ \forall k \in K. \quad (1)$$

This integral converges for $\operatorname{Re} u > 0$ (see [5], Chapter VII or [12], Chapter 10.1).

The mapping $f \mapsto J_u f$ is continuous and the family of operators $\{J_u \mid u \in \mathbb{C}\}$ is holomorphic in u for $\operatorname{Re} u > 0$ with respect to appropriate topologies (see [5], Chapter VII.7 or [12], Chapter 10.1).

For $u \in \mathbb{R}_{>0}$ the operator J_u is self-adjoint with respect to the usual $L^2(K)$ -scalar product.

Moreover, one can extend the function $u \mapsto J_u$ meromorphically to \mathbb{C} (see [12], Chapter 10.1). Then, for every $u \in \mathbb{C}$ for which the operator J_u is regular, it is an intertwining operator from $\mathcal{P}^{+,u}$ to $\mathcal{P}^{+,-u}$.

Remark 4.1.

The operator J_u commutes with the projections p_n for all $n \in \mathbb{N}$ and for every $u \in \mathbb{C}$ for which J_u is regular.

This can be seen as follows: Since J_u is a G -intertwining operator, it is a K -intertwining operator as well and therefore, it leaves $\mathcal{H}_{\mathcal{P}^{+,u}}(n)$, the n -th isotypic component of the representation $\mathcal{P}^{+,u}$ which was defined above, invariant. Hence, one can easily conclude that $p_n \circ J_u = J_u \circ p_n$ and the assertion follows.

One can now deduce that the operators J_u have the property

$$J_u|_{\mathcal{H}_{\mathcal{P}^{+,u}}(n)} = c_n(u) \cdot \text{id}|_{\mathcal{H}_{\mathcal{P}^{+,u}}(n)} \quad \text{for all even } n \in \mathbb{Z}, \quad (2)$$

as an equality of meromorphic functions, where $c_n : \mathbb{C} \rightarrow \mathbb{C}$ is a meromorphic function for every even $n \in \mathbb{N}$. This follows from the above remark together with the fact that $\mathcal{H}_{\mathcal{P}^{+,u}}(n)$ is one-dimensional.

Using standard integral formulas (see [5], Chapter 5.6), one gets by (1) for $u = 1$

$$J_1(f) = c \int_K f(k) dk \quad \forall f \in C^\infty(K)_+$$

for a constant $c > 0$. Therefore and since $\mathcal{H}_{\mathcal{P}^{+,u}}(n) = \mathbb{C} \cdot e^{-in\cdot}$ for every even $n \in \mathbb{N}$ which was stated above, one gets

$$c_n(1) = \begin{cases} \neq 0 & \text{for } n = 0 \\ = 0 & \text{for } n \in \mathbb{Z} \setminus \{0\} \text{ even.} \end{cases} \quad (3)$$

For the convenience of the reader, an explicit formula for the quotients of the functions c_n will now be given. However, this formula will not be used in this article. Instead, further necessary properties of the intertwining operator J_u and the functions c_n will be concluded from the irreducibility of the representations $\mathcal{P}^{\pm,u}$.

Remark 4.2.

The quotients of the functions c_n can be given by

$$\frac{c_n(u)}{c_0(u)} = \frac{(u-1)(u-3) \cdots (u-(|n|-1))}{(u+1)(u+3) \cdots (u+(|n|-1))} \cdot (-1)^{\frac{n}{2}} \quad \text{for } u \in \mathbb{C} \text{ and all even } n \in \mathbb{Z}.$$

This formula can be deduced from a formula for $c_n(u)$ in terms of Gamma functions which can be found in [2], and the Gamma function recurrence formula. Here, one has to remark that the definition of $c_n(u)$ in [2] differs by a sign from its definition in this article.

Next, it can be shown that

$$c_0(u) \neq 0 \quad \text{for } u \in (0, 1) : \quad (4)$$

For this, assume that $c_0(u) = 0$ for an element $u \in (0, 1)$. Then, by (2), the operator J_u has a non-zero kernel. Furthermore, the representation $\mathcal{P}^{+,u}$ is irreducible on $C^\infty(K)_+$ (see Remark 2.4). As $\ker(J_u)$ is a closed invariant subspace with respect to the representation $\mathcal{P}^{+,u}$, the kernel of J_u has to be the whole space $C^\infty(K)_+$ and hence the operator J_u is identically zero. But there is no $u \in \mathbb{C}$ with $\operatorname{Re} u > 0$ such that the operator J_u is identically zero (see [12], Chapter 10.1). Therefore, $c_0(u) \neq 0$ for every $u \in (0, 1)$.

Thus, one can define

$$\tilde{J}_u := \frac{1}{c_0(u)} J_u$$

for $u \in \mathbb{C}$ as a meromorphic function.

Lemma 4.3.

\tilde{J}_u is regular at $u = 0$ and $\tilde{J}_0 = \operatorname{id}$.

Proof:

First, one can observe that on $\mathcal{H}_{\mathcal{P}^{+,u}}(0)$ the operator \tilde{J}_u is always equal to the identity and that in particular $\tilde{J}_0|_{\mathcal{H}_{\mathcal{P}^{+,0}}(0)}$ also equals $\operatorname{id}|_{\mathcal{H}_{\mathcal{P}^{+,0}}(0)}$.

Now, it has to be shown that \tilde{J}_u is regular at $u = 0$.

Since the mapping $u \mapsto \tilde{J}_u$ is an operator-valued meromorphic function for $u \in \mathbb{C}$, one can represent it locally as a Laurent series in 0 with finite principal part and operators as coefficients in the following way:

$$\tilde{J}_u = \sum_{k=k_0}^{\infty} L_k u^k \quad \text{on } C^\infty(K)_+$$

for operators L_k going from $C^\infty(K)_+$ to $C^\infty(K)_+$ for $k \geq k_0$ and where $k_0 \in \mathbb{Z}$ is the smallest number such that $L_{k_0} \neq 0$.

Moreover, this gives

$$L_{k_0} = \lim_{u \rightarrow 0} u^{-k_0} \tilde{J}_u.$$

Now, for the above desired regularity of \tilde{J}_u at $u = 0$, it has to be shown that $k_0 \geq 0$.

So, assume that $k_0 < 0$.

As every \tilde{J}_u is an intertwining operator from $\mathcal{P}^{+,u}$ to $\mathcal{P}^{+,-u}$, L_{k_0} is an intertwining operator from $\mathcal{P}^{+,0}$ to itself, i.e. it commutes with $\mathcal{P}^{+,0}$. Furthermore, L_{k_0} vanishes on $\mathcal{H}_{\mathcal{P}^{+,0}}(0)$, the space of all constant functions on K , because \tilde{J}_u equals the identity on $\mathcal{H}_{\mathcal{P}^{+,u}}(0)$ and thus does not have a pole there.

Moreover, since L_{k_0} commutes with $\mathcal{P}^{+,0}$, it vanishes on $\mathcal{P}^{+,0}(g)\mathcal{H}_{\mathcal{P}^{+,0}}(0)$ for every $g \in G$. In addition, the representation $\mathcal{P}^{+,0}$ is irreducible on the space $C^\infty(K)_+$ (see Remark 2.4). Furthermore, for every $0 \neq \xi \in \mathcal{H}_{\mathcal{P}^{+,0}}(0)$, the subspace $\operatorname{span}\{\mathcal{P}^{+,0}(g)\xi \mid g \in G\}$ is G -invariant and hence, by the irreducibility, $C^\infty(K)_+ \subset \overline{\operatorname{span}\{\mathcal{P}^{+,0}(g)\xi \mid g \in G\}}$. Thus, L_{k_0} vanishes on the whole space $C^\infty(K)_+$, which is a contradiction to the choice of k_0 .

Hence, one gets $k_0 \geq 0$, which means that the mapping $u \mapsto \tilde{J}_u$ does not have any poles in $u = 0$. Therefore, \tilde{J}_u is regular at $u = 0$ on $C^\infty(K)_+$.

Moreover, as above, as a limit of intertwining operators, $\tilde{J}_0 = \lim_{u \rightarrow 0} \tilde{J}_u$ is an intertwining operator, which intertwines the irreducible unitary representation $\mathcal{P}^{+,0}$ with itself (see Remark 2.4 for the irreducibility and the unitarity of $\mathcal{P}^{+,0}$). Hence, by Schur's Lemma \tilde{J}_0 is

a scalar multiple of the identity. Since it equals the identity on $\mathcal{H}_{\mathcal{P}^+,0}(0)$, one gets $\tilde{J}_0 = \text{id}$. \square

From (4) and Lemma 4.3, one can conclude that

$$\frac{c_n(u)}{c_0(u)} > 0 \quad \text{for } u \in (0, 1) \text{ and for all even } n \in \mathbb{Z} : \quad (5)$$

Using the same argumentation for the operator \tilde{J}_u as in the proof of (4), one also gets that $\frac{c_n(u)}{c_0(u)} \neq 0$ for every $u \in (0, 1)$.

Furthermore, from Lemma 4.3 one can deduce that $\frac{c_n(0)}{c_0(0)} = 1$, and with the continuity of $\frac{c_n}{c_0}$ on $(0, 1)$, one gets $\frac{c_n(u)}{c_0(u)} > 0$ for all $u \in (0, 1)$ and all even $n \in \mathbb{Z}$ and (5) is shown.

Now, define a scalar product on $C^\infty(K)_+$ as follows:

$$\langle f_1, f_2 \rangle_u := \langle \tilde{J}_u f_1, f_2 \rangle_{L^2(K)}.$$

Lemma 4.4.

$\langle \cdot, \cdot \rangle_u$ is an invariant positive definite scalar product for $u \in (0, 1)$.

Proof:

$\langle \cdot, \cdot \rangle_u$ is hermitian:

By the definition of \tilde{J}_u and as J_u is self-adjoint with respect to the usual $L^2(K)$ -scalar product for $u \in (0, 1)$, this is straightforward.

$\langle \cdot, \cdot \rangle_u$ is invariant:

For every $g \in G$ the operator $(\mathcal{P}^{+,u}(g))^{-1}$ is the adjoint operator of $\mathcal{P}^{+,-u}(g)$ with respect to the usual $L^2(K)$ -scalar product. Now, let $f_1, f_2 \in C^\infty(K)_+$. Then, as \tilde{J}_u intertwines $\mathcal{P}^{+,u}$ and $\mathcal{P}^{+,-u}$, one gets for every $g \in G$

$$\begin{aligned} \langle \mathcal{P}^{+,u}(g)f_1, \mathcal{P}^{+,u}(g)f_2 \rangle_u &= \langle \tilde{J}_u \circ \mathcal{P}^{+,u}(g)f_1, \mathcal{P}^{+,u}(g)f_2 \rangle_{L^2(K)} \\ &= \langle \mathcal{P}^{+,-u}(g) \circ \tilde{J}_u f_1, \mathcal{P}^{+,u}(g)f_2 \rangle_{L^2(K)} \\ &= \langle \tilde{J}_u f_1, f_2 \rangle_{L^2(K)} = \langle f_1, f_2 \rangle_u. \end{aligned}$$

$\langle \cdot, \cdot \rangle_u$ is positive definite:

As $\frac{c_n(u)}{c_0(u)} > 0$ by (5), one gets for every $n \in \mathbb{Z}$ and $f \in \mathcal{H}_{\mathcal{P}^+,u}(n)$ by (2) above,

$$\langle f, f \rangle_u = \langle \tilde{J}_u f, f \rangle_{L^2(K)} = \frac{c_n(u)}{c_0(u)} \langle \text{id}_{|\mathcal{H}_{\mathcal{P}^+,u}(n)} f, f \rangle_{L^2(K)} = \frac{c_n(u)}{c_0(u)} \langle f, f \rangle_{L^2(K)} \geq 0 \quad \text{and}$$

$$\langle f, f \rangle_u = 0 \iff \langle f, f \rangle_{L^2(K)} = 0 \iff f = 0.$$

Now, since the direct sum $\bigoplus_{n \in \mathbb{Z}} \mathcal{H}_{\mathcal{P}^+,u}(n)$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_u$ and dense in $\mathcal{H}_{\mathcal{P}^+,u} = L^2(K)_+$, these observations hold for all $f \in C^\infty(K)_+$. \square

The completion of $C^\infty(K)_+$ with respect to this scalar product $\langle \cdot, \cdot \rangle_u$ gives a Hilbert space \mathcal{H}_u . Considering the restriction of the representation $\mathcal{P}^{+,u}$ to $C^\infty(K)_+$ and then continuously extending it to the space \mathcal{H}_u , one gets a unitary representation which will be denoted by $\mathcal{P}^{+,u}$ as well. G acts on \mathcal{H}_u by this unitary representation $\mathcal{P}^{+,u}$.

Furthermore, let $d_n(u) := \sqrt{\frac{c_n(u)}{c_0(u)}} > 0$ for $u \in (0, 1)$. Next, a unitary bijection

$$K_u : \mathcal{H}_u \rightarrow L^2(K)_+ \quad \forall u \in (0, 1)$$

shall be defined. On the n -th isotypic component in \mathcal{H}_u , define K_u by

$$K_u|_{\mathcal{H}_{\mathcal{P}^{+,u}}(n)} := d_n(u) \cdot \text{id}|_{\mathcal{H}_{\mathcal{P}^{+,u}}(n)} \quad \text{for all even } n \in \mathbb{Z}.$$

Then one can extend this definition to finite sums of K -types. This operator also is self-adjoint with respect to the usual $L^2(K)$ -scalar product and for finite sums of K -types f_1 and f_2

$$\langle K_u f_1, K_u f_2 \rangle_{L^2(K)} = \langle K_u^2 f_1, f_2 \rangle_{L^2(K)} = \langle \tilde{J}_u f_1, f_2 \rangle_{L^2(K)} = \langle f_1, f_2 \rangle_u.$$

From this, it follows directly that it is unitary (if one regards the space \mathcal{H}_u equipped with $\langle \cdot, \cdot \rangle_u$ and $L^2(K)_+$ with $\langle \cdot, \cdot \rangle_{L^2(K)}$) and hence, because of the density of $\bigoplus_{n \in \mathbb{Z}} \mathcal{H}_{\mathcal{P}^{+,u}}(n)$ in \mathcal{H}_u , one can extend K_u continuously on the whole space \mathcal{H}_u .

Moreover, K_u is continuous in u and one also has the property $\lim_{u \rightarrow 0} K_u = \text{id}$.

By its definition, the operator K_u commutes with the projections p_n for all $n \in \mathbb{N}$ as well.

Now, by [5], Chapter 14.4, one gets the identity

$$J_{-u} \circ J_u = \gamma(u) \cdot \text{id}, \tag{6}$$

where $u \mapsto \gamma(u)$ is a meromorphic function. One can also obtain this equation by observing that $J_{-u} \circ J_u$ is an intertwining operator of the representation $\mathcal{P}^{+,u}$, which is irreducible for almost all $u \in \mathbb{C}$ (see Remark 2.4), with itself, and by using a version of Schur's Lemma for (\mathfrak{g}, K) -modules.

By restricting the above equation to the n -th isotypic component, one gets thus the relation

$$c_n(u)c_n(-u) = \gamma(u) \quad \text{for all even } n \in \mathbb{Z} \tag{7}$$

as meromorphic functions.

Next, another scalar product on the space $C^\infty(K)_{++} := \{f \in C^\infty(K)_+ \mid p_n(f) = 0 \ \forall n \leq 0\}$ is needed:

For this, define

$$\tilde{J}_{(u)} := \frac{1}{c_2(u)} J_u$$

for $u \in \mathbb{C}$ as a meromorphic family of operators. Then, on $\mathcal{H}_{\mathcal{P}^{+,u}}(2)$ the operator $\tilde{J}_{(u)}$ is equal to the identity for every $u \in \mathbb{C}$ and in particular $\tilde{J}_{(1)}|_{\mathcal{H}_{\mathcal{P}^{+,1}}(2)}$ equals $\text{id}|_{\mathcal{H}_{\mathcal{P}^{+,1}}(2)}$.

Moreover, define the space $C^\infty(K)_{+-} := \{f \in C^\infty(K)_+ \mid p_n(f) = 0 \ \forall n \geq 0\}$.

Then, the representation $\mathcal{P}^{+,1}$ is irreducible on the spaces $C^\infty(K)_{++}$ and $C^\infty(K)_{+-}$ (see [11], Chapter 5.6).

Lemma 4.5.

(a) $\tilde{J}_{(-u)} \circ \tilde{J}_{(u)} = \tilde{J}_{(u)} \circ \tilde{J}_{(-u)} = \text{id}$ as a meromorphic family.

(b) $\tilde{J}_{(u)}$ is regular at $u = -1$ and

$$\ker(\tilde{J}_{(-1)}) \cap C^\infty(K)_{++} = \ker(\tilde{J}_{(-1)}) \cap C^\infty(K)_{+-} = \{0\}.$$

Furthermore, $\tilde{J}_{(-1)}$ is an intertwining operator of $\mathcal{P}^{+,-1}$ with $\mathcal{P}^{+,1}$.

(c) $\tilde{J}_{(u)}|_{C^\infty(K)_{++} \oplus C^\infty(K)_{+-}}$ is regular at $u = 1$.

Proof:

One has for all $n \in \mathbb{Z}$ and all $u \in \mathbb{C}$

$$\tilde{J}_{(-u)} \circ \tilde{J}_{(u)}|_{\mathcal{H}_{\mathcal{P}^{+,u}(n)}} \stackrel{(6)}{=} \frac{\gamma(u)}{c_2(u)c_2(-u)} \cdot \text{id}|_{\mathcal{H}_{\mathcal{P}^{+,u}(n)}} \stackrel{(7)}{=} \text{id}|_{\mathcal{H}_{\mathcal{P}^{+,u}(n)}}$$

and thus

$$\tilde{J}_{(-u)} \circ \tilde{J}_{(u)} = \tilde{J}_{(u)} \circ \tilde{J}_{(-u)} = \text{id}$$

for $u \in \mathbb{C}$ as a meromorphic family and (a) follows.

Because the mapping $u \mapsto \tilde{J}_{(u)}$ is an operator-valued meromorphic function for $u \in \mathbb{C}$, one can represent it locally as a Laurent series in -1 with finite principal part and operators as coefficients in the following way:

$$\tilde{J}_{(u)} = \sum_{k=k_0}^{\infty} L_k(u+1)^k \quad \text{on } C^\infty(K)_+$$

for operators L_k going from $C^\infty(K)_+$ to $C^\infty(K)_+$ for $k \geq k_0$ and where $k_0 \in \mathbb{Z}$ is the smallest number such that $L_{k_0} \neq 0$.

Then, one gets

$$L_{k_0} = \lim_{u \rightarrow 1} (-u+1)^{-k_0} \tilde{J}_{(-u)} = \lim_{u \rightarrow 1} (-u+1)^{-k_0} \frac{J_{-u}}{c_2(-u)} \quad (8)$$

and thus, as a limit of intertwining operators, L_{k_0} is an intertwining operator of $\mathcal{P}^{+,-1}$ with $\mathcal{P}^{+,1}$. Now, it shall first be shown that

$$\ker(L_{k_0}) \cap C^\infty(K)_{++} = \ker(L_{k_0}) \cap C^\infty(K)_{+-} = \{0\}. \quad (9)$$

Since $\mathcal{H}_{\mathcal{P}^{+,1}(n)} = \mathcal{H}_{\mathcal{P}^{+,-1}(n)}$,

$$L_{k_0}|_{\mathcal{H}_{\mathcal{P}^{+,1}(n)}} = a_n \cdot \text{id}|_{\mathcal{H}_{\mathcal{P}^{+,1}(n)}} \quad \text{for } a_n := \lim_{u \rightarrow 1} \frac{c_n(-u)}{c_2(-u)} (-u+1)^{-k_0}.$$

As $C^\infty(K)_{++} = \bigoplus_{\substack{n \in \mathbb{Z}_{>0} \\ n \text{ even}}} \mathcal{H}_{\mathcal{P}^{+,1}(n)}$ and $C^\infty(K)_{+-} = \bigoplus_{\substack{n \in \mathbb{Z}_{<0} \\ n \text{ even}}} \mathcal{H}_{\mathcal{P}^{+,1}(n)}$, for (9) it has to be shown

that $a_n \neq 0$ for all even $n \in \mathbb{Z} \setminus \{0\}$.

By (7), one has

$$\frac{c_n(u)c_n(-u)}{c_2(-u)} = \frac{\gamma(u)}{c_2(-u)},$$

i.e. the left hand side does not depend on n . So the order of pole for the limit for $u \rightarrow 1$ has to be the same for every $n \in \mathbb{Z}$. But as $c_n(1) = 0$ for $n \neq 0$ and $c_0(1) \neq 0$ by (3), $c_0(-1)$ has to be 0 in order to get the same order of pole for $n = 0$ and $n \neq 0$. It follows that $a_0 = 0$. As k_0 was chosen in such a way that $L_{k_0} \neq 0$, there has to exist $0 \neq n \in \mathbb{Z}$ with $a_n \neq 0$. Furthermore, from (1), one can conclude that $\overline{J_u f} = J_{\bar{u}} \bar{f}$ for all $u \in \mathbb{C}$ with $\operatorname{Re} u > 0$. By extending meromorphically, this holds for all $u \in \mathbb{C}$ and, by regarding the n -th isotypic component, one can deduce that $\overline{c_n(u)} = c_{-n}(\bar{u})$. Hence, one gets for every $n \in \mathbb{Z}$ and every $u \in \mathbb{R}$ that $\overline{c_n(u)} = c_{-n}(u)$. This means that there also exists an integer $n_1 > 0$ such that $a_{n_1} \neq 0 \neq a_{-n_1}$. Therefore,

$$L_{k_0|_{\mathcal{H}_{\mathcal{P}^{+,1}}(n_1)}} = a_{n_1} \cdot \operatorname{id}|_{\mathcal{H}_{\mathcal{P}^{+,1}}(n_1)} \neq 0 \neq a_{-n_1} \cdot \operatorname{id}|_{\mathcal{H}_{\mathcal{P}^{+,1}}(-n_1)} = L_{k_0|_{\mathcal{H}_{\mathcal{P}^{+,1}}(-n_1)}}$$

and thus

$$\operatorname{Im}\left(L_{k_0|_{\mathcal{H}_{\mathcal{P}^{+,1}}(n_1)}}\right) \subset \mathcal{H}_{\mathcal{P}^{+,1}}(n_1) \subset C^\infty(K)_{++} \quad \text{and}$$

$$\operatorname{Im}\left(L_{k_0|_{\mathcal{H}_{\mathcal{P}^{+,1}}(-n_1)}}\right) \subset \mathcal{H}_{\mathcal{P}^{+,1}}(-n_1) \subset C^\infty(K)_{+-}.$$

So, in particular, $\operatorname{Im}(L_{k_0}) \cap C^\infty(K)_{++} \neq \{0\} \neq \operatorname{Im}(L_{k_0}) \cap C^\infty(K)_{+-}$. But as $\mathcal{P}^{+,1}$ is irreducible on $C^\infty(K)_{++}$ and on $C^\infty(K)_{+-}$, one gets

$$\overline{\operatorname{Im}(L_{k_0}) \cap C^\infty(K)_{++}} = C^\infty(K)_{++} \quad \text{and} \quad \overline{\operatorname{Im}(L_{k_0}) \cap C^\infty(K)_{+-}} = C^\infty(K)_{+-},$$

where the completions are regarded in the spaces $C^\infty(K)_{++}$ and respectively $C^\infty(K)_{+-}$. Hence, $\mathcal{H}_{\mathcal{P}^{+,1}}(n) \subset \operatorname{Im}(L_{k_0})$ for all even $n \neq 0$ and therefore, $a_n \neq 0$ for every even $n \neq 0$ and (9) follows.

One can now conclude that $k_0 = 0$:

Since

$$L_{k_0|_{\mathcal{H}_{\mathcal{P}^{+,1}}(2)}} = \lim_{u \rightarrow 1} (-u + 1)^{-k_0} \cdot \operatorname{id}|_{\mathcal{H}_{\mathcal{P}^{+,1}}(2)}$$

does not have any poles, $k_0 \not\neq 0$, and as it does not have any zeros, $k_0 \not\neq 0$.

Hence,

$$L_{k_0} = \lim_{u \rightarrow 1} \tilde{J}_{(-u)} = \tilde{J}_{(-1)}$$

and thus, (b) follows by (9) and as L_{k_0} intertwines $\mathcal{P}^{+,-1}$ with $\mathcal{P}^{+,1}$.

Now, by (b), $\tilde{J}_{(u)}$ is regular at $u = -1$. As by (a) one has $\tilde{J}_{(-1)} \circ \tilde{J}_{(1)} = \tilde{J}_{(1)} \circ \tilde{J}_{(-1)} = \operatorname{id}$ and since by (b) the operator $\tilde{J}_{(-1)}$ is nowhere equal to 0 on $C^\infty(K)_{++} \oplus C^\infty(K)_{+-}$, the operator $\tilde{J}_{(u)}$ has to be regular on $C^\infty(K)_{++} \oplus C^\infty(K)_{+-}$ at $u = 1$ as well and (c) is shown. \square

Moreover, $\tilde{J}_{(1)}$ is not identically 0 on the space $C^\infty(K)_{++}$, as it equals the identity on $\mathcal{H}_{\mathcal{P}^{+,1}}(2)$. Thus, regard the operator $\tilde{J}_{(1)}$ on the space $C^\infty(K)_{++}$. This operator is injective, since for every $f \in C^\infty(K)_{++}$ with $\tilde{J}_{(1)}(f) = 0$, one gets by Lemma 4.5(a) that $0 = \tilde{J}_{(-1)} \circ \tilde{J}_{(1)}(f) = \operatorname{id}(f) = f$.

Now, define for all functions $f_1, f_2 \in C^\infty(K)_{++}$

$$\langle f_1, f_2 \rangle_{(1)} := \langle \tilde{J}_{(1)} f_1, f_2 \rangle_{L^2(K)}.$$

Furthermore, choose \tilde{f}_1 in such a way that $\tilde{J}_{(-1)}(\tilde{f}_1) = f_1$. Then

$$\langle f_1, f_2 \rangle_{(1)} = \langle \tilde{f}_1, f_2 \rangle_{L^2(K)}, \quad (10)$$

since the scalar product does not depend on the choice of \tilde{f}_1 :

It is possible to add an element $\tilde{f}_{ker} \in \ker(\tilde{J}_{(-1)})$ to the function \tilde{f}_1 . But by Lemma 4.5(b) above, one has $\ker(\tilde{J}_{(-1)}) \cap C^\infty(K)_{++} = \ker(\tilde{J}_{(-1)}) \cap C^\infty(K)_{+-} = \{0\}$ and therefore, $\tilde{f}_{ker} \in \{f \in C^\infty(K)_+ \mid p_n(f) = 0 \ \forall n \neq 0\}$. Thus, the function \tilde{f}_{ker} is orthogonal to $f_2 \in C^\infty(K)_{++} = \{f \in C^\infty(K)_+ \mid p_n(f) = 0 \ \forall n \leq 0\}$ and the scalar product stays the same.

Lemma 4.6.

$\langle \cdot, \cdot \rangle_{(1)}$ is an invariant positive definite scalar product.

Proof:

$\langle \cdot, \cdot \rangle_{(1)}$ is hermitian:

As above for $\langle \cdot, \cdot \rangle_u$, this is straightforward.

$\langle \cdot, \cdot \rangle_{(1)}$ is invariant:

Let $f_1, f_2 \in C^\infty(K)_{++}$ and choose \tilde{f}_1 such that $\tilde{J}_{(-1)}(\tilde{f}_1) = f_1$.

Then, as $\tilde{J}_{(-1)}$ intertwines $\mathcal{P}^{+,-1}$ and $\mathcal{P}^{+,1}$ by Lemma 4.5(b), for every $g \in G$

$$\tilde{J}_{(-1)}(\mathcal{P}^{+,-1}(g)\tilde{f}_1) = \mathcal{P}^{+,1}(g) \circ \tilde{J}_{(-1)}(\tilde{f}_1) = \mathcal{P}^{+,1}(g)f_1.$$

Hence, one can choose

$$\widetilde{\mathcal{P}^{+,1}(g)f_1} := \mathcal{P}^{+,-1}(g)\tilde{f}_1.$$

Since for all $g \in G$ the operator $(\mathcal{P}^{+,1}(g))^{-1}$ is the adjoint operator of $\mathcal{P}^{+,-1}(g)$ with respect to the usual $L^2(K)$ -scalar product, one gets for every $g \in G$

$$\begin{aligned} \langle \mathcal{P}^{+,1}(g)f_1, \mathcal{P}^{+,1}(g)f_2 \rangle_{(1)} &= \left\langle \widetilde{\mathcal{P}^{+,1}(g)f_1}, \mathcal{P}^{+,1}(g)f_2 \right\rangle_{L^2(K)} \\ &= \left\langle \mathcal{P}^{+,-1}(g)\tilde{f}_1, \mathcal{P}^{+,1}(g)f_2 \right\rangle_{L^2(K)} \\ &= \langle \tilde{f}_1, f_2 \rangle_{L^2(K)} \\ &= \langle f_1, f_2 \rangle_{(1)}. \end{aligned}$$

$\langle \cdot, \cdot \rangle_{(1)}$ is positive definite:

$\frac{c_n(u)}{c_2(u)} > 0$ for every $u \in (0, 1)$, since

$$\frac{c_n(u)}{c_2(u)} = \frac{c_n(u)}{c_0(u)} \cdot \frac{c_0(u)}{c_2(u)} > 0$$

by (5) of the beginning of this subsection. Hence, its limit $\lim_{u \rightarrow 1} \frac{c_n(u)}{c_2(u)}$ is larger or equal to 0 as well. Therefore, similar as above for $\langle \cdot, \cdot \rangle_u$, for every $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and for every $f_1 \in \mathcal{H}_{\mathcal{P}^{+,1}(n)}$, one gets

$$\langle f_1, f_1 \rangle_{(1)} = \langle \tilde{J}_{(1)}f_1, f_1 \rangle_{L^2(K)} = \lim_{u \rightarrow 1} \frac{c_n(u)}{c_2(u)} \langle f_1, f_1 \rangle_{L^2(K)} \geq 0$$

from the argument above and since $\mathcal{H}_{\mathcal{P}+,1}(n) = \mathcal{H}_{\mathcal{P}+,u}(n)$ for every $u \in (0,1)$. Moreover, because of the injectivity of $\tilde{J}_{(1)}$

$$f_1 = 0 \iff \tilde{J}_{(1)}f_1 = 0 \iff \lim_{u \rightarrow 1} \frac{c_n(u)}{c_2(u)} f_1 = 0,$$

which means that $\lim_{u \rightarrow 1} \frac{c_n(u)}{c_2(u)} > 0$. Hence,

$$\langle f_1, f_1 \rangle_{(1)} = 0 \iff \langle f_1, f_1 \rangle_{L^2(K)} = 0 \iff f_1 = 0.$$

As the direct sum $\bigoplus_{n \in \mathbb{N}^*} \mathcal{H}_{\mathcal{P}+,1}(n)$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_{(1)}$ and dense in $C^\infty(K)_{++}$, the positive definiteness is shown everywhere. □

Now, the completion of the space $C^\infty(K)_{++}$ with respect to this scalar product gives a Hilbert space which will be called $\mathcal{H}_{(1)}$.

The same procedure can be accomplished for $C^\infty(K)_{+-} = \{f \in C^\infty(K)_+ \mid p_n(f) = 0 \forall n \geq 0\}$: Here, for $u \in (0,1)$, define the operator $\tilde{J}_{[u]}$ as

$$\tilde{J}_{[u]} := \frac{1}{c_{-2}(u)} J_u.$$

As above, it can be shown that $\tilde{J}_{[u]}$ is regular at $u = 1$ on $C^\infty(K)_{+-}$ and is not identically 0 there.

Thus, define again for all $f_1, f_2 \in C^\infty(K)_{+-}$

$$\langle f_1, f_2 \rangle_{[1]} := \langle \tilde{J}_{[1]}f_1, f_2 \rangle_{L^2(K)}.$$

This is another invariant positive definite scalar product. The completion of the space $C^\infty(K)_{+-}$ with respect to this scalar product gives a Hilbert space called $\mathcal{H}_{[1]}$.

4.2 Description of the irreducible unitary representations

Now, some convenient realizations for the unitary dual of $SL(2, \mathbb{R})$ shall be provided.

The unitary dual \widehat{G} of $G = SL(2, \mathbb{R})$ consists of the following representations:

1. The **principal series** representations:

- (a) $\mathcal{P}^{+,iv}$ for $v \in [0, \infty)$.
- (b) $\mathcal{P}^{-,iv}$ for $v \in (0, \infty)$.

See Section 2.2 above for the definitions.

2. The **complementary series** representations \mathcal{C}^u for $u \in (0,1)$:
The Hilbert space $\mathcal{H}_{\mathcal{C}^u}$ is defined by

$$\mathcal{H}_{\mathcal{C}^u} := L^2(K)_+$$

and the action is given by

$$\mathcal{C}^u(g) := K_u \circ \mathcal{P}^{+,u}(g) \circ K_u^{-1}$$

for all $g \in G$, where here again $\mathcal{P}^{+,u}(g)$ is meant in the following way: One considers the restriction of $\mathcal{P}^{+,u}(g)$ to $C^\infty(K)_+$ and then continuously extends it to the space \mathcal{H}_u (see the definition of \mathcal{H}_u in Section 4.1).

3. The **discrete series** representations:

(a) \mathcal{D}_m^+ for odd $m \in \mathbb{N}^*$:

i. \mathcal{D}_1^+ :

The Hilbert space $\mathcal{H}_{\mathcal{D}_1^+}$ is given by

$$\mathcal{H}_{\mathcal{D}_1^+} := \mathcal{H}_{(1)}$$

defined in Section 4.1. The action is given by

$$\mathcal{D}_1^+ := \mathcal{P}^{+,1}.$$

Here again, as well as in all the definitions in this subsection, the representation $\mathcal{P}^{+,u}$ for the different values $u \in \mathbb{C}$ is meant as described above: One restricts it to the respective subspace of $L^2(K)$ and then continuously extends it to the respective Hilbert space.

ii. \mathcal{D}_m^+ for odd $m \in \mathbb{N}_{\geq 3}$:

As a Hilbert space $\mathcal{H}_{\mathcal{D}_m^+}$ for \mathcal{D}_m^+ for odd $m \in \mathbb{N}_{\geq 3}$ one can take the completion of the space

$$\{f \in C^\infty(K)_+ \mid p_n(f) = 0 \ \forall n \leq m-1\}$$

with respect to an appropriate scalar product, and as the action

$$\mathcal{D}_m^+ := \mathcal{P}^{+,m}.$$

With this realization, the Hilbert spaces $\mathcal{H}_{\mathcal{D}_m^+}$ for odd $m \in \mathbb{N}_{\geq 3}$ depend on m . But as all of them are infinite-dimensional and separable, one can identify them if one conjugates the respective G -action. So, fix an infinite-dimensional separable Hilbert space $\mathcal{H}_{\mathcal{D}}$. Furthermore, the G -action is not needed for the determination of $C^*(G)$.

(b) \mathcal{D}_m^- for odd $m \in \mathbb{N}^*$:

i. \mathcal{D}_1^- :

The Hilbert space $\mathcal{H}_{\mathcal{D}_1^-}$ is given by

$$\mathcal{H}_{\mathcal{D}_1^-} := \mathcal{H}_{[1]}$$

defined in Section 4.1 above and the action is given by

$$\mathcal{D}_1^- := \mathcal{P}^{+,1}.$$

ii. \mathcal{D}_m^- for odd $m \in \mathbb{N}_{\geq 3}$:

Similar as for \mathcal{D}_m^+ , as a Hilbert space $\mathcal{H}_{\mathcal{D}_m^-}$ for \mathcal{D}_m^- for odd $m \in \mathbb{N}_{\geq 3}$ one can take the completion of the space

$$\{f \in C^\infty(K)_+ \mid p_n(f) = 0 \ \forall n \geq -m + 1\}$$

with respect to an appropriate scalar product, and as the action

$$\mathcal{D}_m^- := \mathcal{P}^{+,m}.$$

Again, the Hilbert spaces depend on m . One identifies them and takes the common infinite-dimensional separable Hilbert space $\mathcal{H}_{\mathcal{D}}$ fixed in (a)(ii). Again, the G -action is not needed for the determination of $C^*(G)$.

(c) \mathcal{D}_m^+ for even $m \in \mathbb{N}^*$:

As a Hilbert space $\mathcal{H}_{\mathcal{D}_m^+}$ for \mathcal{D}_m^+ for even $m \in \mathbb{N}^*$ one can take the completion of the space

$$\{f \in C^\infty(K)_- \mid p_n(f) = 0 \ \forall n \leq m - 1\}$$

with respect to an appropriate scalar product, and as the action

$$\mathcal{D}_m^+ := \mathcal{P}^{-,m}.$$

Again, the Hilbert spaces are identified and one takes the common infinite-dimensional separable Hilbert space $\mathcal{H}_{\mathcal{D}}$, as in (a)(ii).

(d) \mathcal{D}_m^- for even $m \in \mathbb{N}^*$:

As a Hilbert space $\mathcal{H}_{\mathcal{D}_m^-}$ for \mathcal{D}_m^- for even $m \in \mathbb{N}^*$ one can take the completion of the space

$$\{f \in C^\infty(K)_- \mid p_n(f) = 0 \ \forall n \geq -m + 1\}$$

with respect to an appropriate scalar product, and as the action

$$\mathcal{D}_m^- := \mathcal{P}^{-,m}.$$

Here again, the Hilbert spaces are identified and one takes the common Hilbert space $\mathcal{H}_{\mathcal{D}}$, as in (a)(ii).

4. The limits of the discrete series:

(a) \mathcal{D}_+ :

The Hilbert space $\mathcal{H}_{\mathcal{D}_+}$ is defined by

$$\mathcal{H}_{\mathcal{D}_+} := \{f \in L^2(K)_- \mid p_n(f) = 0 \ \forall n \leq 0\}$$

and the action is given by

$$\mathcal{D}_+ := \mathcal{P}^{-,0}.$$

(b) \mathcal{D}_- :

The Hilbert space $\mathcal{H}_{\mathcal{D}_-}$ is defined by

$$\mathcal{H}_{\mathcal{D}_-} := \{f \in L^2(K)_- \mid p_n(f) = 0 \ \forall n \geq 0\}$$

and the action is given by

$$\mathcal{D}_- := \mathcal{P}^{-,0}.$$

5. The **trivial** representation \mathcal{F}_1 :

Its Hilbert space $\mathcal{H}_{\mathcal{F}_1} = \mathbb{C}$ will be identified with the space of constant functions

$$\{f \in L^2(K)_+ \mid p_n(f) = 0 \ \forall n \neq 0\}.$$

Here, the action is given by

$$\mathcal{F}_1(g) := \text{id}$$

for all $g \in G$. One also has

$$\mathcal{F}_1 = \mathcal{P}^{+, -1},$$

as $\nu_{-1} + \rho = 0$ and every $f \in \mathcal{H}_{\mathcal{F}_1}$ is a constant function.

A discussion of all irreducible representations of $SL(2, \mathbb{R})$ without scalar products can be found in [11], Chapter 5.6. But as the scalar products in this article are shown to be unitary and invariant, they are the correct ones.

In [5], Chapter 2.5, a different realization of the discrete series representations can be read that is not used in this article. An alternative description of all irreducible unitary representations can be found in [6], Chapter 6.6.

Remark 4.7.

$$\mathcal{H}_{\mathcal{P}^-, 0} \cong \mathcal{H}_{\mathcal{D}^+} \oplus \mathcal{H}_{\mathcal{D}^-}.$$

Remark 4.8.

By the definition of the operator K_u by means of its value on the space $\mathcal{H}_{\mathcal{P}^+, u}$, one can easily verify that p_n also projects $\mathcal{H}_{\mathcal{C}^u}$ to its n -th isotypic component $\mathcal{H}_{\mathcal{C}^u}(n)$ for every $n \in \mathbb{Z}$.

Furthermore, for every irreducible unitary representation π of G the operator p_n leaves the above defined Hilbert space \mathcal{H}_π invariant, since all of the Hilbert spaces are completions of the space $C^\infty(K)$ fulfilling p_n -cancelation properties for certain $n \in \mathbb{Z}$.

Hence, for every $n \in \mathbb{Z}$ and for every irreducible unitary representation π of G , the operator p_n is the projection of \mathcal{H}_π to the n -th isotypic component $\mathcal{H}_\pi(n)$.

4.3 The $SL(2, \mathbb{R})$ -representations applied to the Casimir operator

Regard the Casimir operator \mathcal{C} in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ with respect to the non-degenerate symmetric and Ad -invariant bilinear form on \mathfrak{g} defined by

$$\langle X, Y \rangle := 2 \operatorname{tr}(XY) \quad \forall X, Y \in \mathfrak{g}.$$

In order to be able to describe the topology on \widehat{G} , the above listed representations applied to the Casimir operator will now be given:

Lemma 4.9.

Applied to the Casimir operator, the representations $\mathcal{P}^{\pm, u}$ for $u \in \mathbb{C}$ give the following:

$$\mathcal{P}^{\pm, u}(\mathcal{C}) = \frac{1}{4}(u^2 - 1) \cdot \text{id}.$$

The proof of this Lemma is standard and consists of an easy calculation using the fact that $\mathcal{P}^{\pm, u}(g) \circ \mathcal{P}^{\pm, u}(\mathcal{C}) = \mathcal{P}^{\pm, u}(\mathcal{C}) \circ \mathcal{P}^{\pm, u}(g)$ for all $g \in G$. (Compare [11], Chapter 5.6.)

From Lemma 4.9 one can deduce for the above listed irreducible unitary representations of $SL(2, \mathbb{R})$:

1. $\mathcal{P}^{\pm,iv}(\mathcal{C}) = \frac{1}{4}(-v^2 - 1) \cdot \text{id} \quad \forall v \in [0, \infty),$
2. $\mathcal{C}^u(\mathcal{C}) = K_u \circ \frac{1}{4}(u^2 - 1) \cdot \text{id} \circ K_u^{-1} = \frac{1}{4}(u^2 - 1) \cdot \text{id} \quad \forall u \in (0, 1),$
3. $\mathcal{D}_m^{\pm}(\mathcal{C}) = \frac{1}{4}(m^2 - 1) \cdot \text{id} \quad \forall m \in \mathbb{N}^*,$
4. $\mathcal{D}_{\pm}(\mathcal{C}) = \frac{1}{4}(0 - 1) \cdot \text{id} = -\frac{1}{4} \cdot \text{id} \quad \text{and}$
5. $\mathcal{F}_1(\mathcal{C}) = \frac{1}{4}((-1)^2 - 1) \cdot \text{id} = 0.$

4.4 The topology on $\widehat{SL(2, \mathbb{R})}$

With the help of the above computations, it is now possible to describe the topology on \widehat{G} :

Proposition 4.10.

The topology on \widehat{G} can be characterized in the following way:

1. *For all sequences $(v_j)_{j \in \mathbb{N}}$ and all v in $[0, \infty)$*

$$\mathcal{P}^{+,iv_j} \xrightarrow{j \rightarrow \infty} \mathcal{P}^{+,iv} \iff v_j \xrightarrow{j \rightarrow \infty} v.$$

2. *For all sequences $(v_j)_{j \in \mathbb{N}}$ and all v in $(0, \infty)$*

$$\mathcal{P}^{-,iv_j} \xrightarrow{j \rightarrow \infty} \mathcal{P}^{-,iv} \iff v_j \xrightarrow{j \rightarrow \infty} v.$$

For all sequences $(v_j)_{j \in \mathbb{N}}$ in $[0, \infty)$

$$\mathcal{P}^{-,iv_j} \xrightarrow{j \rightarrow \infty} \{\mathcal{D}_+, \mathcal{D}_-\} \iff v_j \xrightarrow{j \rightarrow \infty} 0.$$

3. *For all sequences $(u_j)_{j \in \mathbb{N}}$ and all u in $(0, 1)$*

$$\mathcal{C}^{u_j} \xrightarrow{j \rightarrow \infty} \mathcal{C}^u \iff u_j \xrightarrow{j \rightarrow \infty} u.$$

For all sequences $(u_j)_{j \in \mathbb{N}}$ in $(0, 1)$

$$\mathcal{C}^{u_j} \xrightarrow{j \rightarrow \infty} \mathcal{P}^{+,0} \iff u_j \xrightarrow{j \rightarrow \infty} 0.$$

For all sequences $(u_j)_{j \in \mathbb{N}}$ in $(0, 1)$

$$\mathcal{C}^{u_j} \xrightarrow{j \rightarrow \infty} \{\mathcal{D}_1^+, \mathcal{D}_1^-, \mathcal{F}_1\} \iff u_j \xrightarrow{j \rightarrow \infty} 1.$$

All other sequences $(\pi_j)_{j \in \mathbb{N}}$ can only converge if they fulfill one of the following conditions:

- (a) *They are constant for large $j \in \mathbb{N}$. In that case they have one single limit point, namely the value they take for large j .*

- (b) For large $j \in \mathbb{N}$ they are one of the above listed sequences. Then they converge in the above described way.
- (c) For large $j \in \mathbb{N}$ the sequence consists only of members \mathcal{P}^{+,iv_j} and \mathcal{C}^{u_j} . Then it converges to $\mathcal{P}^{+,0}$ if and only if $v_j \xrightarrow{j \rightarrow \infty} 0$ as well as $u_j \xrightarrow{j \rightarrow \infty} 0$ (compare 1. for $v = 0$ and the second part of 3. above).
- (d) For large $j \in \mathbb{N}$ the sequence consists only of members \mathcal{C}^{u_j} and \mathcal{D}_1^+ . Then it converges to \mathcal{D}_1^+ if and only if $u_j \xrightarrow{j \rightarrow \infty} 1$ (compare the third part of 3. above).
- (e) For large $j \in \mathbb{N}$ the sequence consists only of members \mathcal{C}^{u_j} and \mathcal{D}_1^- . Then it converges to \mathcal{D}_1^- if and only if $u_j \xrightarrow{j \rightarrow \infty} 1$ (compare the third part of 3. above).
- (f) For large $j \in \mathbb{N}$ the sequence consists only of members \mathcal{C}^{u_j} and \mathcal{F}_1 . Then it converges to \mathcal{F}_1 if and only if $u_j \xrightarrow{j \rightarrow \infty} 1$ (compare the third part of 3. above).
- (g) For large $j \in \mathbb{N}$ the sequence consists only of members \mathcal{P}^{-,iv_j} and \mathcal{D}_+ . Then it converges to \mathcal{D}_+ if and only if $v_j \xrightarrow{j \rightarrow \infty} 0$ (compare the second part of 2. above).
- (h) For large $j \in \mathbb{N}$ the sequence consists only of members \mathcal{P}^{-,iv_j} and \mathcal{D}_- . Then it converges to \mathcal{D}_- if and only if $v_j \xrightarrow{j \rightarrow \infty} 0$ (compare the second part of 2. above).

Since the topology of \widehat{G} for $G = SL(2, \mathbb{R})$ is well-known, the proof of this proposition will be a bit sketchy, containing the ideas but leaving out most of the calculations:

If a sequence of representations $(\pi_j)_{j \in \mathbb{N}}$ converges to a representation π , the sequence $(\pi_j(\mathcal{C}))_{j \in \mathbb{N}}$ has to converge to $\pi(\mathcal{C})$ as well. By the observations of Section 4.3, the left hand side thus implies the right hand side in all the cases.

Now, for the other implication, it will first be shown, that for a sequence $(u_j)_{j \in \mathbb{N}}$ in \mathbb{C} , $u \in \mathbb{C}$ and for every compact set $\tilde{K} \subset G$

$$u_j \xrightarrow{j \rightarrow \infty} u \implies \sup_{g \in \tilde{K}} \|\mathcal{P}^{\pm, u_j}(g) - \mathcal{P}^{\pm, u}(g)\|_{op} \xrightarrow{j \rightarrow \infty} 0 : \quad (11)$$

One has

$$\begin{aligned} & \sup_{g \in \tilde{K}} \|\mathcal{P}^{\pm, u_j}(g) - \mathcal{P}^{\pm, u}(g)\|_{op}^2 \\ &= \sup_{g \in \tilde{K}} \sup_{\substack{f \in L^2(K)_{\pm} \\ \|f\|_2 = 1}} \int_K |\mathcal{P}^{\pm, u_j}(g)(f)(k) - \mathcal{P}^{\pm, u}(g)(f)(k)|^2 dk \\ &= \sup_{g \in \tilde{K}} \sup_{\substack{f \in L^2(K)_{\pm} \\ \|f\|_2 = 1}} \int_K \left| e^{-\nu_{u_j} H(g^{-1}k)} - e^{-\nu_u H(g^{-1}k)} \right|^2 e^{-2\rho H(g^{-1}k)} \left| f(\kappa(g^{-1}k)) \right|^2 dk. \end{aligned}$$

Now, for $g \in \tilde{K}$ and $k \in K$, $g^{-1}k$ is also contained in a compact set and therefore, $H(g^{-1}k)$ is contained in a compact set as well. As

$$H(g^{-1}k) = \begin{pmatrix} h(g^{-1}k) & 0 \\ 0 & -h(g^{-1}k) \end{pmatrix} \quad \text{for } h(g^{-1}k) \in \mathbb{R},$$

there is thus a compact set $I \subset \mathbb{R}$ such that $h(g^{-1}k) \in I$ for all $g \in \tilde{K}$ and all $k \in K$. Hence

$$\begin{aligned}
& \sup_{g \in \tilde{K}} \sup_{\substack{f \in L^2(K)_\pm \\ \|f\|_2=1}} \int_K \left| e^{-\nu_{u_j} H(g^{-1}k)} - e^{-\nu_u H(g^{-1}k)} \right|^2 e^{-2\rho H(g^{-1}k)} \left| f(\kappa(g^{-1}k)) \right|^2 dk \\
& \leq \sup_{x \in I} \left| e^{-u_j x} - e^{-ux} \right|^2 \sup_{g \in \tilde{K}} \sup_{\substack{f \in L^2(K)_\pm \\ \|f\|_2=1}} \int_K e^{-2\rho H(g^{-1}k)} \left| f(\kappa(g^{-1}k)) \right|^2 dk \\
& \stackrel{\text{Lemma 2.6}}{=} \sup_{x \in I} \left| e^{-u_j x} - e^{-ux} \right|^2 \sup_{\substack{f \in L^2(K)_\pm \\ \|f\|_2=1}} \|f\|_{L^2(K)}^2 \\
& = \sup_{x \in I} \left| e^{-u_j x} - e^{-ux} \right|^2 \xrightarrow{j \rightarrow \infty} 0.
\end{aligned}$$

Therefore, (11) follows.

From this, one can easily deduce that for a sequence $(u_j)_{j \in \mathbb{N}}$ in \mathbb{C} and $u \in \mathbb{C}$

$$u_j \xrightarrow{j \rightarrow \infty} u \implies \mathcal{P}^{\pm, u_j} \xrightarrow{j \rightarrow \infty} \mathcal{P}^{\pm, u}$$

in the sense of convergence of matrix coefficients described in Theorem 2.3, i.e. for some $f \in \mathcal{H}_{\mathcal{P}^{\pm, u}}$ there exists for every $j \in \mathbb{N}$ a function $f_j \in \mathcal{H}_{\mathcal{P}^{\pm, u_j}}$ such that

$$\langle \mathcal{P}^{\pm, u_j}(\cdot) f_j, f_j \rangle_{\mathcal{H}_{\mathcal{P}^{\pm, u_j}}} \xrightarrow{j \rightarrow \infty} \langle \mathcal{P}^{\pm, u}(\cdot) f, f \rangle_{\mathcal{H}_{\mathcal{P}^{\pm, u}}} \quad (12)$$

uniformly on compacta.

This yields directly the second implication of 1. and 2. (using that $\mathcal{H}_{\mathcal{D}_+}, \mathcal{H}_{\mathcal{D}_-} \subset \mathcal{H}_{\mathcal{P}^-, iv}$ for all $v \in (0, \infty)$).

Furthermore, one gets the equality

$$\langle K_u \circ \mathcal{P}^{+, u}(g) \circ K_u^{-1} f, f \rangle_{L^2(K)} = \langle \mathcal{P}^{+, u}(g) f, f \rangle_{L^2(K)} \quad (13)$$

for $\tilde{n} \in \mathbb{Z}$, $u \in (0, 1)$, $g \in G$ and $f \in \mathcal{H}_{\mathcal{P}^{+, u}(\tilde{n})}$.

In order to show this, one uses the fact that the operator K_u is self-adjoint with respect to the usual $L^2(K)$ -scalar product.

Now, by (13), choosing f in the space $\mathcal{H}_{\mathcal{P}^{\pm, u}(n)}$ for the matching value $n \in \mathbb{Z}$, one can express the matrix coefficients $\langle \pi(g) f, f \rangle_{\mathcal{H}_\pi}$ for every representation $\pi \in \widehat{G}$ as $\langle \mathcal{P}^{+, u}(g) f, f \rangle_{L^2(K)}$. Then, one gets by (12) for $f_j = f$ the convergences needed for the second implications of 3., using Lemma 4.5 and the characterization of the $\mathcal{H}_{(1)}$ -scalar product in (10).

At the end it still has to be shown that these are all possibilities of convergence.

For this, first, one can see that for a sequence of representations $(\pi_j)_{j \in \mathbb{N}}$ converging to a representation π , there has to be a common K -type for $(\pi_j)_{j \text{ large}}$ and π . With this fact and by considering the values the representations of $SL(2, \mathbb{R})$ take on the Casimir operator (see Section 4.3), one can separate several sets of representations from each other such that only the possibilities of convergence listed in this proposition are possible. \square

4.5 Definition of subsets Γ_i of the unitary dual

Now, the unitary dual will be divided into different subsets which are thereafter proved to meet the requirements of the norm controlled dual limit conditions:

Define

$$\begin{aligned}
\Gamma_0 &:= \{\mathcal{F}_1\} \\
\Gamma_1 &:= \{\mathcal{D}_1^+\} \\
\Gamma_2 &:= \{\mathcal{D}_1^-\} \\
\Gamma_3 &:= \{\mathcal{D}_+\} \\
\Gamma_4 &:= \{\mathcal{D}_-\} \\
\Gamma_5 &:= \{\mathcal{D}_m^\pm \mid m \in \mathbb{N}_{>1}\} \\
\Gamma_6 &:= \{\mathcal{P}^{+,iv} \mid v \in [0, \infty)\} \\
\Gamma_7 &:= \{\mathcal{P}^{-,iv} \mid v \in (0, \infty)\} \\
\Gamma_8 &:= \{\mathcal{C}^u \mid u \in (0, 1)\}.
\end{aligned}$$

Obviously, all the sets Γ_i for $i \in \{0, \dots, 8\}$ are Hausdorff. Furthermore, the sets

$$S_i := \bigcup_{j \in \{0, \dots, i\}} \Gamma_j$$

are closed and the set $S_0 = \Gamma_0$ consists of all the characters of $G = SL(2, \mathbb{R})$. In addition, as defined in Section 4.2, for every $i \in \{0, \dots, 8\}$, there exists one common Hilbert space \mathcal{H}_i that all the representations in Γ_i act on.

Therefore, Condition 1 of Definition 3.1 is fulfilled.

As every semisimple Lie group with a finite center meets the CCR-condition, Condition 2 of Definition 3.1 is fulfilled as well. Thus, Condition 3 remains to be shown:

5 Condition 3(a)

For the proof of Condition 3(a), as well as for the proof of Condition 3(b), some preliminaries are needed:

Lemma 5.1.

Let $i \in \{0, \dots, 8\}$, let M be a dense subset of $C^*(G)$, let $\tilde{\nu} : CB(S_{i-1}) \rightarrow \mathcal{B}(\mathcal{H}_i)$ be a linear map bounded by $c\|\cdot\|_{S_{i-1}}$ for a constant $c > 0$ and let $(\pi_j)_{j \in \mathbb{N}}$ be representations in Γ_i such that

$$\left\| \pi_j(h) - \tilde{\nu}(\mathcal{F}(h)|_{S_{i-1}}) \right\|_{op} \xrightarrow{j \rightarrow \infty} 0 \quad \forall h \in M.$$

Then

$$\left\| \pi_j(a) - \tilde{\nu}(\mathcal{F}(a)|_{S_{i-1}}) \right\|_{op} \xrightarrow{j \rightarrow \infty} 0 \quad \forall a \in C^*(G).$$

Proof:

Let $a \in C^*(G)$ and $\varepsilon > 0$.

Since M is dense in $C^*(G)$, there exists $h \in M$ such that $\|h - a\|_{C^*(G)} < \frac{\varepsilon}{3\tilde{c}}$ for $\tilde{c} := \max\{1, c\}$. Furthermore, there exists $J(\varepsilon) > 0$ in such a way that for all integers $j \geq J(\varepsilon)$ one has $\left\| \pi_j(h) - \tilde{\nu}(\mathcal{F}(h)|_{S_{i-1}}) \right\|_{op} < \frac{\varepsilon}{3}$.

Then, for all $j \geq J(\varepsilon)$

$$\begin{aligned} \left\| \pi_j(a) - \tilde{\nu}(\mathcal{F}(a)|_{S_{i-1}}) \right\|_{op} &\leq \left\| \pi_j(a) - \pi_j(h) \right\|_{op} + \left\| \pi_j(h) - \tilde{\nu}(\mathcal{F}(h)|_{S_{i-1}}) \right\|_{op} \\ &\quad + \left\| \tilde{\nu}(\mathcal{F}(h)|_{S_{i-1}}) - \tilde{\nu}(\mathcal{F}(a)|_{S_{i-1}}) \right\|_{op}. \end{aligned}$$

By assumption, $\left\| \pi_j(h) - \tilde{\nu}(\mathcal{F}(h)|_{S_{i-1}}) \right\|_{op} < \frac{\varepsilon}{3}$. In addition, as π_j is a homomorphism

$$\left\| \pi_j(a) - \pi_j(h) \right\|_{op} = \left\| \pi_j(a - h) \right\|_{op} \leq \sup_{\tilde{\pi} \in \widehat{G}} \left\| \tilde{\pi}(a - h) \right\|_{op} = \|a - h\|_{C^*(G)} < \frac{\varepsilon}{3\tilde{c}}.$$

Moreover,

$$\begin{aligned} \left\| \tilde{\nu}(\mathcal{F}(h)|_{S_{i-1}}) - \tilde{\nu}(\mathcal{F}(a)|_{S_{i-1}}) \right\|_{op} &= \left\| \tilde{\nu}(\mathcal{F}(h)|_{S_{i-1}} - \mathcal{F}(a)|_{S_{i-1}}) \right\|_{op} \\ &\leq c \left\| \mathcal{F}(h)|_{S_{i-1}} - \mathcal{F}(a)|_{S_{i-1}} \right\|_{S_{i-1}} \\ &= c \left\| \mathcal{F}(h - a)|_{S_{i-1}} \right\|_{S_{i-1}} \\ &= c \sup_{\tilde{\pi} \in S_{i-1}} \left\| \mathcal{F}(h - a)(\tilde{\pi}) \right\|_{op} \\ &\leq c \sup_{\tilde{\pi} \in \widehat{G}} \left\| \tilde{\pi}(h - a) \right\|_{op} \\ &= c \|h - a\|_{C^*(G)} < \frac{\varepsilon}{3}. \end{aligned}$$

Thus,

$$\left\| \pi_j(a) - \tilde{\nu}(\mathcal{F}(a)|_{S_{i-1}}) \right\|_{op} < \varepsilon$$

and the claim is shown. □

Define the $K \times K$ -representation $\pi_{K \times K}$ on the space $V_{\pi_{K \times K}} := C_0^\infty(G)$ of compactly supported $C^\infty(G)$ -functions as

$$\begin{aligned} \pi_{K \times K} &: K \times K \rightarrow \mathcal{B}(C_0^\infty(G)), \\ \pi_{K \times K}(k_1, k_2)h(g) &:= h(k_1^{-1}gk_2) \quad \forall (k_1, k_2) \in K \times K \quad \forall h \in C_0^\infty(G) \quad \forall g \in G. \end{aligned}$$

For all $l, n \in \mathbb{Z}$

$$\begin{aligned} V_{\pi_{K \times K}}(l, n) &= \left\{ h \in C_0^\infty(G) \mid \pi_{K \times K}(k_{\varphi_1}, k_{\varphi_2})h = e^{il\varphi_1 + in\varphi_2}h \quad \forall \varphi_1, \varphi_2 \in [0, 2\pi) \right\} \\ &= \left\{ h \in C_0^\infty(G) \mid h(k_{\varphi_1}^{-1}gk_{\varphi_2}) = e^{il\varphi_1 + in\varphi_2}h(g) \quad \forall \varphi_1, \varphi_2 \in [0, 2\pi) \right\}. \end{aligned}$$

Then the algebraic direct sum $\bigoplus_{l, n \in \mathbb{Z}} V_{\pi_{K \times K}}(l, n)$ is dense in $V_{\pi_{K \times K}} = C_0^\infty(G)$ with respect to the $L^1(G)$ -norm and as $\|\cdot\|_{C^*(G)} \leq \|\cdot\|_{L^1(G)}$ on $L^1(G)$, it is dense with respect to the

$C^*(G)$ -norm as well. $C_0^\infty(G)$ in turn is dense in $C^*(G)$. Hence, the algebraic direct sum $\bigoplus_{l,n \in \mathbb{Z}} V_{\pi_{K \times K}}(l, n)$ is also dense in $C^*(G)$.

Let $p_{l,n}$ the projection going from $V_{\pi_{K \times K}}$ to $V_{\pi_{K \times K}}(l, n)$ defined in the following way:

For $h \in V_{\pi_{K \times K}} = C_0^\infty(G)$ and $g \in G$

$$p_{l,n}(h)(g) := \frac{1}{|K|^2} \int_{K \times K} h(k_{\varphi_1} g k_{\varphi_2}^{-1}) e^{il\varphi_1} e^{in\varphi_2} d(k_{\varphi_1}, k_{\varphi_2}).$$

Similar as shown in Lemma 5.1 above, in order to prove that a sequence of representations applied to general elements $a \in C^*(G)$ converges to a representation applied to a , it suffices to show this convergence for elements h in a dense subset $M \subset C^*(G)$. Therefore and with Lemma 5.1, due to the density discussed above, instead of dealing with a general $a \in C^*(G)$, the calculations in Sections 5 and 6 can be accomplished with a function $h \in C_0^\infty(G)$ fulfilling $h = p_{l,-n}(h)$ for some integers $l, n \in \mathbb{Z}$.

Lemma 5.2.

For $f \in L^2(K)_+$ and $h \in C_0^\infty(G)$ with $h = p_{l,-n}(h)$ for $l, n \in \mathbb{Z}$, one gets

$$\mathcal{P}^{+,u}(h)(f) = \mathcal{P}^{+,u}(h)(p_n(f)) = p_l(\mathcal{P}^{+,u}(h)(p_n(f))) \quad \forall u \in \mathbb{C}.$$

Furthermore, if $h \neq 0$ the integers l and n must be even.

This lemma is a standard formula. Its proof consists of a simple calculation and will thus be skipped.

Now, the proof of Condition 3(a) can be executed.

Condition 3(a) is obvious for the sets Γ_i for $i \in \{0, \dots, 5\}$, as these are discrete sets.

So, let $v_j, v \in [0, \infty)$ and $(\mathcal{P}^{+,iv_j})_{j \in \mathbb{N}}$ a sequence in Γ_6 converging to $\mathcal{P}^{+,iv}$. Then $v_j \xrightarrow{j \rightarrow \infty} v$. Hence, let $h \in C_0^\infty(G)$ be supported in the compact set $\tilde{K} \in G$. Then, very similar as in the proof of (11), there is a compact set $I \subset \mathbb{R}$ such that

$$\begin{aligned} & \|\mathcal{P}^{+,iv_j}(h) - \mathcal{P}^{+,iv}(h)\|_{op}^2 \\ &= \sup_{\substack{f \in L^2(K)_+ \\ \|f\|_2=1}} \int_K \left| \int_{\tilde{K}} h(g) \left(\mathcal{P}^{+,iv_j}(g)(f)(k) - \mathcal{P}^{+,iv}(g)(f)(k) \right) dg \right|^2 dk \\ &\leq \sup_{x \in I} \left| e^{-iv_j x} - e^{-iv x} \right|^2 \sup_{\substack{f \in L^2(K)_+ \\ \|f\|_2=1}} \int_K \left(\int_{\tilde{K}} |h(g)| e^{-\rho H(g^{-1}k)} \left| f(\kappa(g^{-1}k)) \right| dg \right)^2 dk \\ &\stackrel{\text{H\"older}}{\leq} \sup_{x \in I} \left| e^{-iv_j x} - e^{-iv x} \right|^2 \|h\|_{L^2(G)}^2 \sup_{\substack{f \in L^2(K)_+ \\ \|f\|_2=1}} \int_K \int_{\tilde{K}} e^{-2\rho H(g^{-1}k)} \left| f(\kappa(g^{-1}k)) \right|^2 dg dk \\ &\stackrel{\text{Lemma 2.6}}{=} |\tilde{K}| \sup_{x \in I} \left| e^{-iv_j x} - e^{-iv x} \right|^2 \|h\|_{L^2(G)}^2 \xrightarrow{j \rightarrow \infty} 0, \end{aligned} \tag{14}$$

as $v_j \xrightarrow{j \rightarrow \infty} v$.

Because of the density of $C_0^\infty(G)$ in $C^*(G)$, one gets the desired convergence for $a \in C^*(G)$.

The reasoning is the same for Γ_7 .

For Γ_8 , let $u_j, u \in (0, 1)$ and $(\mathcal{C}^{u_j})_{j \in \mathbb{N}}$ a sequence in Γ_8 converging to $\mathcal{C}^u \in \Gamma_8$. Then $u_j \xrightarrow{j \rightarrow \infty} u$. Moreover, let $h \in C^*(G)$ and let $l, n \in \mathbb{Z}$ such that $h = p_{l, -n}(h)$, as discussed at the beginning of this section.

Let $f \in L^2(K)_+$ with $\|f\|_{L^2(K)} = 1$.

Since $K_{\tilde{u}}^{-1}$ commutes with p_n , by Lemma 5.2 one has for every $\tilde{u} \in (0, 1)$

$$\begin{aligned} \mathcal{C}^{\tilde{u}}(h)(f) &= K_{\tilde{u}} \circ \mathcal{P}^{+, \tilde{u}}(h) \circ K_{\tilde{u}}^{-1}(f) = K_{\tilde{u}} \circ p_l \left(\mathcal{P}^{+, \tilde{u}}(h) \left(p_n(K_{\tilde{u}}^{-1}(f)) \right) \right) \\ &= d_l(\tilde{u}) p_l \left(\mathcal{P}^{+, \tilde{u}}(h) \left(K_{\tilde{u}}^{-1}(p_n(f)) \right) \right) = \frac{d_l(\tilde{u})}{d_n(\tilde{u})} p_l \left(\mathcal{P}^{+, \tilde{u}}(h) (p_n(f)) \right) \\ &= \sqrt{\frac{c_l(\tilde{u})}{c_n(\tilde{u})}} \mathcal{P}^{+, \tilde{u}}(h)(f). \end{aligned} \quad (15)$$

Hence,

$$\|\mathcal{C}^{u_j}(h) - \mathcal{C}^u(h)\|_{op}^2 = \left\| \sqrt{\frac{c_l(u_j)}{c_n(u_j)}} \mathcal{P}^{+, u_j}(h) - \sqrt{\frac{c_l(u)}{c_n(u)}} \mathcal{P}^{+, u}(h) \right\|_{op}^2 \xrightarrow{j \rightarrow \infty} 0, \quad (16)$$

since $u_j \xrightarrow{j \rightarrow \infty} u$ and with the same reasoning as in (14).

Because of the density of $C_0^\infty(G)$ in $C^*(G)$, one gets the desired convergence for $a \in C^*(G)$ and thus, the claim is also shown for Γ_8 .

6 Condition 3(b)

Now, only Condition 3(b) remains to be shown. This is the most complicated part of the proof of the different conditions listed in Definition 3.1.

The setting of a sequence $(\gamma_j)_{j \in \mathbb{N}}$ in Γ_i converging to a limit set contained in $S_{i-1} = \bigcup_{l < i} \Gamma_l$ regarded in Condition 3(b) can only occur in the following cases:

- (i) $(\gamma_j)_{j \in \mathbb{N}} = (\mathcal{P}^{-, i v_j})_{j \in \mathbb{N}}$ is a sequence in Γ_7 whose limit set is $\Gamma_3 \cup \Gamma_4 = \{\mathcal{D}_+, \mathcal{D}_-\}$.
- (ii) $(\gamma_j)_{j \in \mathbb{N}} = (\mathcal{C}^{u_j})_{j \in \mathbb{N}}$ is a sequence in Γ_8 whose limit set is $\{\mathcal{P}^{+, 0}\} \subset \Gamma_7$.
- (iii) $(\gamma_j)_{j \in \mathbb{N}} = (\mathcal{C}^{u_j})_{j \in \mathbb{N}}$ is a sequence in Γ_8 whose limit set is $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 = \{\mathcal{F}_1, \mathcal{D}_1^+, \mathcal{D}_1^-\}$.

For $i \in \{0, \dots, 6\}$ the sets Γ_i are closed and thus the regarded situation cannot appear for sequences $(\gamma_j)_{j \in \mathbb{N}}$ in Γ_i for $i \in \{0, \dots, 6\}$.

Since all the sequences regarded in the cases (i), (ii) and (iii) are properly converging, the transition to subsequences will be omitted.

First, regard Case (i):

Let $(\mathcal{P}^{-,iv_j})_{j \in \mathbb{N}}$ be a sequence in Γ_7 whose limit set is $\{\mathcal{D}_+, \mathcal{D}_-\}$. As $\mathcal{P}^{-,iv_j} \xrightarrow{j \rightarrow \infty} \{\mathcal{D}_+, \mathcal{D}_-\}$, it follows that $v_j \xrightarrow{j \rightarrow \infty} 0$.

Now, a bounded, linear and involutive mapping

$$\tilde{v}_j : CB(S_6) \rightarrow \mathcal{B}(L^2(K)_-)$$

fulfilling

$$\lim_{j \rightarrow \infty} \left\| \mathcal{P}^{-,iv_j}(a) - \tilde{v}_j(\mathcal{F}(a)|_{S_6}) \right\|_{op} = 0 \quad \forall a \in C^*(G)$$

has to be defined.

Since in this and in the following cases this mapping will not depend on j , it will from now on be denoted by \tilde{v} instead of \tilde{v}_j .

Let p_+ be the projection from $L^2(K)_-$ to the space $\mathcal{H}_{\mathcal{D}_+}$ and p_- the projection from $L^2(K)_-$ to $\mathcal{H}_{\mathcal{D}_-}$. Then, by Remark 4.7 one has $L^2(K)_- = \mathcal{H}_{\mathcal{D}_+} \oplus \mathcal{H}_{\mathcal{D}_-}$, i.e. $\text{id}_{L^2(K)_-} = p_+ + p_-$.

Now, let

$$\tilde{v}(\psi) := \tilde{v}_{\{\mathcal{D}_+, \mathcal{D}_-\}}(\psi) := \psi(\mathcal{D}_+) \circ p_+ + \psi(\mathcal{D}_-) \circ p_- \quad \forall \psi \in CB(S_6).$$

This is well-defined, as $\mathcal{D}_+, \mathcal{D}_- \in S_6$, and furthermore $\tilde{v}(\psi) \in \mathcal{B}(L^2(K)_-)$.

The linearity of the mapping \tilde{v} is clear. For the involutivity, let $\psi \in CB(S_6)$. Then, since p_+ and p_- equal the identity on the image of $\psi(\mathcal{D}_+)$ and respectively $\psi(\mathcal{D}_-)$,

$$\begin{aligned} (\tilde{v}(\psi))^* &= \left(\psi(\mathcal{D}_+) \circ p_+ \right)^* + \left(\psi(\mathcal{D}_-) \circ p_- \right)^* = \left(p_+ \circ \psi(\mathcal{D}_+) \circ p_+ \right)^* + \left(p_- \circ \psi(\mathcal{D}_-) \circ p_- \right)^* \\ &= p_+^* \circ \psi^*(\mathcal{D}_+) \circ p_+^* + p_-^* \circ \psi^*(\mathcal{D}_-) \circ p_-^* = p_+ \circ \psi^*(\mathcal{D}_+) \circ p_+ + p_- \circ \psi^*(\mathcal{D}_-) \circ p_- \\ &= \psi^*(\mathcal{D}_+) \circ p_+ + \psi^*(\mathcal{D}_-) \circ p_- = \tilde{v}(\psi^*). \end{aligned}$$

To show that \tilde{v} is bounded, again let $\psi \in CB(S_6)$:

$$\begin{aligned} \|\tilde{v}(\psi)\|_{op} &= \|\psi(\mathcal{D}_+) \circ p_+ + \psi(\mathcal{D}_-) \circ p_-\|_{op} = \max \left\{ \|\psi(\mathcal{D}_+)\|_{op}, \|\psi(\mathcal{D}_-)\|_{op} \right\} \\ &\leq \sup_{\gamma \in S_6} \|\psi(\gamma)\|_{op} = \|\psi\|_{S_6}. \end{aligned}$$

Now, only the demanded convergence remains to be shown:

For $h \in C_0^\infty(G)$, one has

$$\begin{aligned} \left\| \mathcal{P}^{-,iv_j}(h) - \tilde{v}(\mathcal{F}(h)|_{S_6}) \right\|_{op}^2 &= \left\| \mathcal{P}^{-,iv_j}(h) - \left(\mathcal{F}(h)(\mathcal{D}_+) \circ p_+ + \mathcal{F}(h)(\mathcal{D}_-) \circ p_- \right) \right\|_{op}^2 \\ &= \left\| \mathcal{P}^{-,iv_j}(h) - \left(\mathcal{D}_+(h) \circ p_+ + \mathcal{D}_-(h) \circ p_- \right) \right\|_{op}^2 \\ &= \left\| \mathcal{P}^{-,iv_j}(h) - \left(\mathcal{P}^{-,0}(h) \circ p_+ + \mathcal{P}^{-,0}(h) \circ p_- \right) \right\|_{op}^2 \\ &= \left\| \mathcal{P}^{-,iv_j}(h) - \mathcal{P}^{-,0}(h) \right\|_{op}^2 \xrightarrow{j \rightarrow \infty} 0 \end{aligned}$$

as in (14) and since $v_j \xrightarrow{j \rightarrow \infty} 0$.

Again, because of the density of $C_0^\infty(G)$ in $C^*(G)$ and with Lemma 5.1, one gets the desired convergence for $a \in C^*(G)$.

Now, regard Case (ii):

Let $(\mathcal{C}^{u_j})_{j \in \mathbb{N}}$ be a sequence in Γ_8 whose limit set is $\{\mathcal{P}^{+,0}\}$. Thus, $u_j \xrightarrow{j \rightarrow \infty} 0$. Here, a bounded, linear and involutive mapping

$$\tilde{\nu} : CB(S_7) \rightarrow \mathcal{B}(L^2(K)_+)$$

fulfilling

$$\lim_{j \rightarrow \infty} \left\| \mathcal{C}^{u_j}(a) - \tilde{\nu}(\mathcal{F}(a)|_{S_7}) \right\|_{op} = 0 \quad \forall a \in C^*(G)$$

is needed:

Define

$$\tilde{\nu}(\psi) := \tilde{\nu}_{\mathcal{P}^{+,0}}(\psi) := \psi(\mathcal{P}^{+,0}) \quad \forall \psi \in CB(S_7).$$

$\tilde{\nu}(\psi) \in \mathcal{B}(L^2(K)_+)$ for every $\psi \in CB(S_7)$ and $\tilde{\nu}$ is well-defined, as $\mathcal{P}^{+,0} \in S_7$.

The linearity and the involutivity of $\tilde{\nu}$ are clear.

For the boundedness of $\tilde{\nu}$, let $\psi \in CB(S_7)$. Then

$$\|\tilde{\nu}(\psi)\|_{op} = \|\psi(\mathcal{P}^{+,0})\|_{op} \leq \sup_{\gamma \in S_7} \|\psi(\gamma)\|_{op} = \|\psi\|_{S_7}.$$

Again, it remains to show the demanded convergence:

So, let $h \in C_0^\infty(G)$. Then one can assume again that there exist $l, n \in \mathbb{Z}$ such that $h = p_{l,-n}(h)$.

Since $\lim_{u \rightarrow 0} \frac{c_{\tilde{n}}(u)}{c_{n'}(u)} = 1$ for all $\tilde{n}, n' \in \mathbb{Z}$ by (2) and Lemma 4.3, one gets with (15)

$$\begin{aligned} \left\| \mathcal{C}^{u_j}(h) - \tilde{\nu}(\mathcal{F}(h)|_{S_7}) \right\|_{op}^2 &= \left\| \sqrt{\frac{c_l(u_j)}{c_n(u_j)}} \mathcal{P}^{+,u_j}(h) - \mathcal{F}(h)(\mathcal{P}^{+,0}) \right\|_{op}^2 \\ &= \left\| \sqrt{\frac{c_l(u_j)}{c_n(u_j)}} \mathcal{P}^{+,u_j}(h) - \mathcal{P}^{+,0}(h) \right\|_{op}^2 \xrightarrow{j \rightarrow \infty} 0, \end{aligned}$$

since $u_j \xrightarrow{j \rightarrow \infty} 0$ and with the same arguments as in the section above.

The desired convergence for $a \in C^*(G)$ follows.

Last, regard Case (iii):

Let $(\mathcal{C}^{u_j})_{j \in \mathbb{N}}$ be a sequence in Γ_8 whose limit set is $\{\mathcal{F}_1, \mathcal{D}_1^+, \mathcal{D}_1^-\}$. This means that $u_j \xrightarrow{j \rightarrow \infty} 1$.

Again, a bounded, linear and involutive mapping

$$\tilde{\nu} : CB(S_7) \rightarrow \mathcal{B}(L^2(K)_+)$$

fulfilling

$$\lim_{j \rightarrow \infty} \left\| \mathcal{C}^{u_j}(a) - \tilde{\nu}(\mathcal{F}(a)|_{S_7}) \right\|_{op} = 0 \quad \forall a \in C^*(G)$$

is needed:

For this, let p_+ the projection from $L^2(K)_+$ to the space $\{f \in L^2(K)_+ \mid p_n(f) = 0 \ \forall n \leq 0\}$ and p_- the projection from $L^2(K)_+$ to $\{f \in L^2(K)_+ \mid p_n(f) = 0 \ \forall n \geq 0\}$. Then, since

$$\begin{aligned} L^2(K)_+ &= \{f \in L^2(K)_+ \mid p_n(f) = 0 \ \forall n \leq 0\} + \{f \in L^2(K)_+ \mid p_n(f) = 0 \ \forall n \geq 0\} \\ &\quad + \{f \in L^2(K)_+ \mid p_n(f) = 0 \ \forall n \neq 0\}, \end{aligned}$$

every $f \in L^2(K)_+$ can be written as $f = p_+(f) + p_-(f) + p_0(f)$.
Furthermore, let

$$d_{n,2}(1) := \lim_{u \rightarrow 1} \sqrt{\frac{c_n(u)}{c_2(u)}} \quad \text{for all even } n > 0 \quad \text{and}$$

$$d_{n,-2}(1) := \lim_{u \rightarrow 1} \sqrt{\frac{c_n(u)}{c_{-2}(u)}} \quad \text{for all even } n < 0.$$

The existence of these limits follows with Lemma 4.5(c) in Section 4.1 and the analog statement for $\tilde{J}_{[1]}$.

Now, define the operators

$$K_{(1)} : \mathcal{H}_{(1)} \rightarrow L^2(K)_+ \quad \text{by} \quad K_{(1)|_{\mathcal{H}_{\mathcal{P}^+,1}(n)}} := d_{n,2}(1) \cdot \text{id}_{|\mathcal{H}_{\mathcal{P}^+,1}(n)} \quad \text{for all even } n > 0$$

and

$$K_{[1]} : \mathcal{H}_{[1]} \rightarrow L^2(K)_+ \quad \text{by} \quad K_{[1]|_{\mathcal{H}_{\mathcal{P}^+,1}(n)}} := d_{n,-2}(1) \cdot \text{id}_{|\mathcal{H}_{\mathcal{P}^+,1}(n)} \quad \text{for all even } n < 0.$$

By definition, these operators are linear and they are unitary by the construction of the scalar products defined on $\mathcal{H}_{(1)}$ and $\mathcal{H}_{[1]}$. Moreover, like $\tilde{J}_{(1)}$ and $\tilde{J}_{(-1)}$, they are injective and, as proved in Remark 4.1 for the operators J_u , $K_{(1)}$ and $K_{[1]}$ commute with the projections p_n for all $n \in \mathbb{N}$.

Moreover, one can easily see that

$$K_{(1)}(\mathcal{H}_{(1)}) = \{f \in L^2(K)_+ \mid p_n(f) = 0 \ \forall n \leq 0\} = p_+(L^2(K)_+) \quad \text{and}$$

$$K_{[1]}(\mathcal{H}_{[1]}) = \{f \in L^2(K)_+ \mid p_n(f) = 0 \ \forall n \geq 0\} = p_-(L^2(K)_+).$$

Hence, one can build the inverse of the operators $K_{(1)}$ and $K_{[1]}$ on the image of p_+ or respectively p_- . Therefore, one can define

$$\begin{aligned} \tilde{\nu}(\psi) &:= \tilde{\nu}_{\{\mathcal{F}_1, \mathcal{D}_1^+, \mathcal{D}_1^-\}}(\psi) := K_{(1)} \circ \psi(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ + K_{[1]} \circ \psi(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- \\ &\quad + \psi(\mathcal{F}_1) \circ p_0 \quad \forall \psi \in CB(S_7). \end{aligned}$$

The mapping $\tilde{\nu}$ is well-defined, since $\mathcal{D}_1^+, \mathcal{D}_1^-, \mathcal{F}_1 \in S_7$, and $\tilde{\nu}(\psi) \in \mathcal{B}(L^2(K)_+)$ for every $\psi \in CB(S_7)$. In addition, its linearity is clear again.

For the involutivity and the boundedness, let $\psi \in CB(S_7)$. Then, as $K_{(1)}$ and $K_{[1]}$ are unitary and as p_+ , p_- and p_0 equal the identity on the image of $K_{(1)}$, $K_{[1]}$ and respectively $\psi(\mathcal{F}_1)$,

$$\begin{aligned} &(\tilde{\nu}(\psi))^* \\ &= \left(K_{(1)} \circ \psi(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ + K_{[1]} \circ \psi(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- + \psi(\mathcal{F}_1) \circ p_0 \right)^* \\ &= \left(p_+ \circ K_{(1)} \circ \psi(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ + p_- \circ K_{[1]} \circ \psi(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- + p_0 \circ \psi(\mathcal{F}_1) \circ p_0 \right)^* \\ &= p_+ \circ K_{(1)} \circ \psi^*(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ + p_- \circ K_{[1]} \circ \psi^*(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- + p_0 \circ \psi^*(\mathcal{F}_1) \circ p_0 \\ &= \tilde{\nu}(\psi^*). \end{aligned}$$

Furthermore, since $\|K_{(1)}\|_{op}\|K_{(1)}^{-1}\|_{op} = \|K_{[1]}\|_{op}\|K_{[1]}^{-1}\|_{op} = 1$, one gets

$$\begin{aligned}
\|\tilde{\nu}(\psi)\|_{op} &= \left\| K_{(1)} \circ \psi(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ + K_{[1]} \circ \psi(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- + \psi(\mathcal{F}_1) \circ p_0 \right\|_{op} \\
&= \max \left\{ \left\| K_{(1)} \circ \psi(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \right\|_{op}, \left\| K_{[1]} \circ \psi(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \right\|_{op}, \left\| \psi(\mathcal{F}_1) \right\|_{op} \right\} \\
&= \max \left\{ \left\| \psi(\mathcal{D}_1^+) \right\|_{op}, \left\| \psi(\mathcal{D}_1^-) \right\|_{op}, \left\| \psi(\mathcal{F}_1) \right\|_{op} \right\} \\
&\leq \sup_{\gamma \in S_7} \|\psi(\gamma)\|_{op} = \|\psi\|_{S_7}.
\end{aligned}$$

For the demanded convergence, let $h \in C_0^\infty(G)$. As above in the proof of (ii) and Condition 3(a), one can assume that there exist $l, n \in \mathbb{Z}$ such that $h = p_{l,-n}(h)$.

Let $f \in L^2(K)_+$ with $\|f\|_{L^2(K)} = 1$.

Since $K_{(1)}^{-1}$ and $K_{[1]}^{-1}$ commute with p_n , similar as in the proof of (15), by Lemma 5.2 one gets

$$\begin{aligned}
&\tilde{\nu}(\mathcal{F}(h)|_{S_7})(f) \\
&= K_{(1)} \circ \mathcal{F}(h)(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+(f) + K_{[1]} \circ \mathcal{F}(h)(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_-(f) + \mathcal{F}(h)(\mathcal{F}_1) \circ p_0(f) \\
&= K_{(1)} \circ \mathcal{D}_1^+(h) \circ K_{(1)}^{-1} \circ p_+(f) + K_{[1]} \circ \mathcal{D}_1^-(h) \circ K_{[1]}^{-1} \circ p_-(f) + \mathcal{F}_1(h) \circ p_0(f) \\
&= K_{(1)} \circ \mathcal{P}^{+,1}(h) \circ K_{(1)}^{-1} \circ p_+(f) + K_{[1]} \circ \mathcal{P}^{+,1}(h) \circ K_{[1]}^{-1} \circ p_-(f) + \mathcal{P}^{+,-1}(h) \circ p_0(f) \\
&= K_{(1)} \circ p_l \left(\mathcal{P}^{+,1}(h) \left(p_n \left(K_{(1)}^{-1} \circ p_+(f) \right) \right) \right) + K_{[1]} \circ p_l \left(\mathcal{P}^{+,1}(h) \left(p_n \left(K_{[1]}^{-1} \circ p_-(f) \right) \right) \right) \\
&\quad + p_l \left(\mathcal{P}^{+,-1}(h) \left(p_n(p_0(f)) \right) \right) \\
&= d_{l,2}(1) p_l \left(\mathcal{P}^{+,1}(h) \left(K_{(1)}^{-1} \left(p_n \circ p_+(f) \right) \right) \right) + d_{l,-2}(1) p_l \left(\mathcal{P}^{+,1}(h) \left(K_{[1]}^{-1} \left(p_n \circ p_-(f) \right) \right) \right) \\
&\quad + p_l \left(\mathcal{P}^{+,-1}(h) \left(p_n \circ p_0(f) \right) \right).
\end{aligned}$$

Now, there are three cases to consider: In the first case $n > 0$, in the second case $n < 0$ and in the third case $n = 0$.

So, first let $n > 0$. Then

$$\begin{aligned}
\tilde{\nu}(\mathcal{F}(h)|_{S_7})(f) &= d_{l,2}(1) p_l \left(\mathcal{P}^{+,1}(h) \left(K_{(1)}^{-1} \left(p_n \circ p_+(f) \right) \right) \right) \\
&= \frac{d_{l,2}(1)}{d_{n,2}(1)} p_l \left(\mathcal{P}^{+,1}(h) (p_n(f)) \right) = \lim_{u \rightarrow 1} \sqrt{\frac{c_l(u)}{c_n(u)}} \mathcal{P}^{+,1}(h)(f).
\end{aligned}$$

Therefore, joining this result with (15), one gets

$$\left\| \mathcal{C}^{u_j}(h) - \tilde{\nu}(\mathcal{F}(h)|_{S_7}) \right\|_{op}^2 = \left\| \sqrt{\frac{c_l(u_j)}{c_n(u_j)}} \mathcal{P}^{+,u_j}(h) - \lim_{u \rightarrow 1} \sqrt{\frac{c_l(u)}{c_n(u)}} \mathcal{P}^{+,1}(h) \right\|_{op}^2 \xrightarrow{j \rightarrow \infty} 0,$$

since $u_j \xrightarrow{j \rightarrow \infty} 1$ and with the same reasoning as in (14).

The proof for the case $n < 0$ is the same as the one for the first case, hence it only remains to regard the case $n = 0$:

Since by (3)

$$\frac{c_l(1)}{c_0(1)} = \begin{cases} 1 & \text{if } l = 0 \\ 0 & \text{if } l \neq 0, \end{cases}$$

one gets for $l \neq 0$ with (15)

$$\left\| \mathcal{C}^{u_j}(h) - \tilde{\nu}(\mathcal{F}(h)|_{S_7}) \right\|_{op}^2 = \left\| \sqrt{\frac{c_l(u_j)}{c_0(u_j)}} \mathcal{P}^{+,u_j}(h) - 0 \right\|_{op}^2 \xrightarrow{j \rightarrow \infty} 0,$$

since $u_j \xrightarrow{j \rightarrow \infty} 1$.

So, let $l = 0$ and define $C_h := \int_G h(g) dg$.

Then $\mathcal{F}_1(h) = C_h \cdot \text{id}$.

Furthermore, with Lemma 2.6, one gets for $g \in G$

$$\begin{aligned} p_0 \circ \mathcal{P}^{+,1}(g) \circ p_0(f) &= \frac{1}{|K|} \int_K \mathcal{P}^{+,1}(g) p_0(f)(k) dk \\ &= \frac{p_0(f)}{|K|} \int_K e^{-2\rho H(g^{-1}k)} dk \\ &= \frac{p_0(f)}{|K|} \int_K e^{-2\rho H(g^{-1}k)} \left| 1\left(\kappa(g^{-1}k)\right) \right|^2 dk \\ &= \frac{p_0(f)}{|K|} \|1\|_{L^2(K)} = p_0(f). \end{aligned}$$

Thus, for h one has

$$p_0 \circ \mathcal{P}^{+,1}(h) \circ p_0(f) = \int_G h(g) p_0(f) dg = C_h p_0(f) = \mathcal{F}_1(h) \circ p_0(f).$$

Therefore, for $l = n = 0$, by (15)

$$\left\| \mathcal{C}^{u_j}(h) - \tilde{\nu}(\mathcal{F}(h)|_{S_7}) \right\|_{op}^2 = \left\| p_0 \circ \mathcal{P}^{+,u_j}(h) \circ p_0 - p_0 \circ \mathcal{P}^{+,1}(h) \circ p_0 \right\|_{op}^2 \xrightarrow{j \rightarrow \infty} 0,$$

since $u_j \xrightarrow{j \rightarrow \infty} 1$.

Hence the claim is also shown in the case $n = 0$ and thus

$$\left\| \mathcal{C}^{u_j}(h) - \tilde{\nu}(\mathcal{F}(h)|_{S_7}) \right\|_{op}^2 \xrightarrow{j \rightarrow \infty} 0,$$

as demanded.

Again, the desired convergence for $a \in C^*(G)$ follows.

□

7 Result

Having now verified all the conditions listed in Section 3, Theorem 3.2 is proved and the C^* -algebra of $G = SL(2, \mathbb{R})$ can therefore be characterized as in Theorem 3.4 with the sets Γ_i and S_i and the Hilbert spaces \mathcal{H}_i for $i \in \{0, \dots, 8\}$ defined in Section 4.5 and Section 4.2 and the mappings $\tilde{\nu}$ constructed in Section 6.

Let for a topological Hausdorff space V and a C^* -algebra B , $C_\infty(V, B)$ be the C^* -algebra of all continuous functions defined on V with values in B that are vanishing at infinity. Then, from Theorem 3.4, one can deduce more concretely the following result for $G = SL(2, \mathbb{R})$:

Theorem 7.1.

Let the operator p_+ be the projection from $L^2(K)_\pm$ to $\{f \in L^2(K)_\pm \mid p_n(f) = 0 \ \forall n \leq 0\}$, p_- the projection from $L^2(K)_\pm$ to the space $\{f \in L^2(K)_\pm \mid p_n(f) = 0 \ \forall n \geq 0\}$ and p_0 the projection from $L^2(K)_+$ to $\{f \in L^2(K)_+ \mid p_n(f) = 0 \ \forall n \neq 0\} = \mathbb{C}$.

Then the C^* -algebra $C^*(G)$ of $G = SL(2, \mathbb{R})$ is isomorphic to the direct sum of C^* -algebras

$$\begin{aligned} & \left\{ F \in C_\infty\left(i[0, \infty) \cup [0, 1], \mathcal{K}(L^2(K)_+)\right) \mid F(1) \text{ commutes with } p_+, p_- \text{ and } p_0 \right\} \\ \oplus & \left\{ F \in C_\infty\left(i[0, \infty), \mathcal{K}(L^2(K)_-)\right) \mid F(0) \text{ commutes with } p_+ \text{ and } p_- \right\} \\ \oplus & C_\infty\left(\mathbb{Z} \setminus \{-1, 0, 1\}, \mathcal{K}(\mathcal{H}_{\mathcal{D}})\right) \end{aligned}$$

for the infinite-dimensional and separable Hilbert space $\mathcal{H}_{\mathcal{D}}$ fixed in Section 4.2.

Proof:

The unitary dual of $C^*(G)$ or respectively of G is given by the disjoint union

$$\widehat{G}_{\text{even}} \dot{\cup} \widehat{G}_{\text{odd}} \dot{\cup} \widehat{G}_{\text{discrete}},$$

where the set $\widehat{G}_{\text{even}}$ consists of the even representations $\mathcal{P}^{+,iv}$ for $v \in [0, \infty)$, \mathcal{C}^u for $u \in (0, 1)$, \mathcal{D}_1^+ , \mathcal{D}_1^- and \mathcal{F}_1 , the set \widehat{G}_{odd} consists of the odd representations $\mathcal{P}^{-,iv}$ for $v \in (0, \infty)$, \mathcal{D}_+ and \mathcal{D}_- and the set $\widehat{G}_{\text{discrete}}$ consists of the even or respectively odd representations \mathcal{D}_m^+ for $m \in \mathbb{N}_{>1}$ and \mathcal{D}_m^- for $m \in \mathbb{N}_{>1}$.

These three listed sets of representations are topologically separated from each other (see Section 4.4).

Mapping $\mathcal{P}^{+,iv}$ for $v \in [0, \infty)$ to iv , \mathcal{C}^u for $u \in (0, 1)$ to u and \mathcal{D}_1^+ , \mathcal{D}_1^- and \mathcal{F}_1 to 1, one gets a surjection from $\widehat{G}_{\text{even}}$ to the set $i[0, \infty) \cup [0, 1] =: I_1$.

Furthermore, mapping $\mathcal{P}^{-,iv}$ for $v \in (0, \infty)$ to iv and \mathcal{D}_+ and \mathcal{D}_- to 0, one gets a surjection from \widehat{G}_{odd} to $i[0, \infty) =: I_2$.

Last, mapping \mathcal{D}_m^+ for $m \in \mathbb{N}_{>1}$ to m and \mathcal{D}_m^- for $m \in \mathbb{N}_{>1}$ to $-m$, one gets a surjection from $\widehat{G}_{\text{discrete}}$ to $\mathbb{Z} \setminus \{-1, 0, 1\} =: I_3$.

Hence, one regards the three sets

$$I_1 = i[0, \infty) \cup [0, 1], \quad I_2 = i[0, \infty) \quad \text{and} \quad I_3 = \mathbb{Z} \setminus \{-1, 0, 1\}.$$

In order to prove this theorem, it has to be shown that for every operator field $\varphi = \mathcal{F}(a)$

for $a \in C^*(G)$ that fulfills the properties listed in Theorem 3.4, there exists a mapping $F_a^1 \in \left\{ F \in C_\infty\left(i[0, \infty) \cup [0, 1], \mathcal{K}(L^2(K)_+)\right) \mid F(1) \text{ commutes with } p_+, p_- \text{ and } p_0 \right\} =: P_1$,

a mapping $F_a^2 \in \left\{ F \in C_\infty\left(i[0, \infty), \mathcal{K}(L^2(K)_-)\right) \mid F(0) \text{ commutes with } p_+ \text{ and } p_- \right\} =: P_2$

and a mapping $F_a^3 \in C_\infty\left(\mathbb{Z} \setminus \{-1, 0, 1\}, \mathcal{K}(\mathcal{H}_D)\right) =: P_3$.

On the other hand, for every $F_1 \in P_1$, every $F_2 \in P_2$ and every $F_3 \in P_3$ one has to construct an operator field φ_{F_1, F_2, F_3} over \widehat{G} that fulfills the properties of Theorem 3.4. Since the above mentioned sets of representations \widehat{G}_{even} , \widehat{G}_{odd} and $\widehat{G}_{discrete}$ are topologically separated from each other, it suffices thus to define three different operator fields φ_{F_1} over \widehat{G}_{even} , φ_{F_2} over \widehat{G}_{odd} and φ_{F_3} over $\widehat{G}_{discrete}$.

For every $a \in C^*(G)$ define a function $F_a^1 : I_1 \rightarrow \mathcal{B}(L^2(K)_+)$ by

$$\begin{aligned} F_a^1(x) &:= \mathcal{F}(a)(\mathcal{P}^{+,x}) \quad \forall x \in i[0, \infty), \\ F_a^1(x) &:= \mathcal{F}(a)(\mathcal{C}^x) \quad \forall x \in (0, 1) \quad \text{and} \\ F_a^1(1) &:= K_{(1)} \circ \mathcal{F}(a)(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ + K_{[1]} \circ \mathcal{F}(a)(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- + \mathcal{F}(a)(\mathcal{F}_1) \circ p_0. \end{aligned}$$

By Property 1. of Theorem 3.4, $F_a^1(x) \in \mathcal{K}(L^2(K)_+)$ for all $x \in I_1 \setminus \{1\}$. Moreover, since $\mathcal{F}(a)(\mathcal{D}_1^+)$, $\mathcal{F}(a)(\mathcal{D}_1^-)$ and $\mathcal{F}(a)(\mathcal{F}_1)$ are also compact, their composition with the bounded operators $K_{(1)}$, $K_{(1)}^{-1}$, $K_{[1]}$, $K_{[1]}^{-1}$, p_+ , p_- and p_0 is compact as well. Therefore, $F_a^1(1) \in \mathcal{K}(L^2(K)_+)$ too.

By Property 4. of Theorem 3.4, F_a^1 vanishes at infinity. Moreover, for all $x \in I_1 \setminus \{0, 1\}$, F_a^1 is obviously continuous in x .

For the continuity in 0, let $\bar{u} = (u_j)_{j \in \mathbb{N}}$ be a sequence in $(0, 1)$ converging to 0. Then,

$$\begin{aligned} \lim_{j \rightarrow \infty} \|F_a^1(u_j) - F_a^1(0)\|_{op} &= \lim_{j \rightarrow \infty} \|\mathcal{F}(a)(\mathcal{C}^{u_j}) - \mathcal{F}(a)(\mathcal{P}^{+,0})\|_{op} \\ &= \lim_{j \rightarrow \infty} \|\mathcal{C}^{u_j}(a) - \mathcal{P}^{+,0}(a)\|_{op} \\ &= \lim_{j \rightarrow \infty} \|K_{u_j} \circ \mathcal{P}^{+,u_j}(a) \circ K_{u_j}^{-1} - \text{id} \circ \mathcal{P}^{+,0}(a) \circ \text{id}^{-1}\|_{op} = 0, \end{aligned}$$

as $K_{u_j} \xrightarrow{j \rightarrow \infty} \text{id}$ and with the same arguments as in the proof of (16) in Section 5.

For the continuity in 1, let $\bar{u} = (u_j)_{j \in \mathbb{N}}$ a sequence in $(0, 1)$ converging to 1. By Property 5. of Theorem 3.4 and since $F_a^1(1) = \tilde{\nu}_{\bar{u},j}(\mathcal{F}(a)) = \tilde{\nu}(\mathcal{F}(a))$ by the definition of $\tilde{\nu}$ in Case (iii) of Section 6,

$$\lim_{j \rightarrow \infty} \|F_a^1(u_j) - F_a^1(1)\|_{op} = \lim_{j \rightarrow \infty} \|\mathcal{F}(a)(\mathcal{C}^{u_j}) - \tilde{\nu}(\mathcal{F}(a))\|_{op} = 0.$$

Hence, F_a^1 is also continuous in 0 and in 1 and thus $F_a^1 \in C_\infty\left(i[0, \infty) \cup [0, 1], \mathcal{K}(L^2(K)_+)\right)$. Since p_+ , p_- and p_0 equal the identity on the image of $K_{(1)}$, $K_{[1]}$ and respectively $\mathcal{F}(a)(\mathcal{F}_1)$, as discovered in Section 6, and since $p_+ \circ p_- = p_- \circ p_+ = p_+ \circ p_0 = p_0 \circ p_+ = p_- \circ p_0 = p_0 \circ p_- = 0$, $F_a^1(1)$ commutes with p_+ , p_- and p_0 .

On the other hand, taking a function $F_1 \in C_\infty\left(i[0, \infty) \cup [0, 1], \mathcal{K}(L^2(K)_+)\right)$ that commutes with p_+ , p_- and p_0 , it has to be shown that there is an operator field φ_{F_1} over

\widehat{G}_{even} that meets the Properties 1. to 5. of Theorem 3.4:
Define

$$\begin{aligned}\varphi_{F_1}(\mathcal{P}^{+,x}) &:= F_1(x) \in \mathcal{B}(L^2(K)_+) \quad \forall x \in i[0, \infty), \\ \varphi_{F_1}(\mathcal{C}^x) &:= F_1(x) \in \mathcal{B}(L^2(K)_+) \quad \forall x \in (0, 1), \\ \varphi_{F_1}(\mathcal{D}_1^+) &:= K_{(1)}^{-1} \circ p_+ \circ F_1(1) \circ K_{(1)} \in \mathcal{B}(\mathcal{H}_{(1)}), \\ \varphi_{F_1}(\mathcal{D}_1^-) &:= K_{[1]}^{-1} \circ p_- \circ F_1(1) \circ K_{[1]} \in \mathcal{B}(\mathcal{H}_{[1]}) \quad \text{and} \\ \varphi_{F_1}(\mathcal{F}_1) &:= p_0 \circ F_1(1) \circ p_0 \in \mathbb{C}.\end{aligned}$$

By the definition of $C_\infty(i[0, \infty) \cup [0, 1], \mathcal{K}(L^2(K)_+))$ and as the composition of the compact operator $F_1(1)$ with bounded operators is compact again, the Properties 1. to 4. are obviously fulfilled.

For Property 5., there are two cases to consider: a sequence in $(0, 1)$ converging to 0 and a sequence in $(0, 1)$ converging to 1.

So, first let $(u_j)_{j \in \mathbb{N}}$ be a sequence in $(0, 1)$ converging to 0. Then, by the definition of $\tilde{\nu}$ in Case (ii) of Section 6

$$\|\varphi_{F_1}(\mathcal{C}^{u_j}) - \tilde{\nu}(\varphi_{F_1})\|_{op} = \|F_1(u_j) - \varphi_{F_1}(\mathcal{P}^{+,0})\|_{op} = \|F_1(u_j) - F_1(0)\|_{op} \xrightarrow{j \rightarrow \infty} 0,$$

since F_1 is continuous in 0.

Now, let $(u_j)_{j \in \mathbb{N}}$ a sequence in $(0, 1)$ converging to 1. Then, as $F_1(1)$ commutes with p_+ , p_- and p_0 , by the definition of $\tilde{\nu}$ in Case (iii) of Section 6 and since $p_+ + p_- + p_0 = \text{id}_{L^2(K)_+ \rightarrow L^2(K)_+}$,

$$\begin{aligned}& \|\varphi_{F_1}(\mathcal{C}^{u_j}) - \tilde{\nu}(\varphi_{F_1})\|_{op} \\ &= \left\| F_1(u_j) - K_{(1)} \circ \varphi_{F_1}(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ - K_{[1]} \circ \varphi_{F_1}(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- - \varphi_{F_1}(\mathcal{F}_1) \circ p_0 \right\|_{op} \\ &= \left\| F_1(u_j) - K_{(1)} \circ K_{(1)}^{-1} \circ p_+ \circ F_1(1) \circ K_{(1)} \circ K_{(1)}^{-1} \circ p_+ \right. \\ & \quad \left. - K_{[1]} \circ K_{[1]}^{-1} \circ p_- \circ F_1(1) \circ K_{[1]} \circ K_{[1]}^{-1} \circ p_- - p_0 \circ F_1(1) \circ p_0 \circ p_0 \right\|_{op} \\ &= \left\| F_1(u_j) - p_+ \circ F_1(1) \circ p_+ - p_- \circ F_1(1) \circ p_- - p_0 \circ F_1(1) \circ p_0 \right\|_{op} \\ &= \left\| F_1(u_j) - F_1(1) \circ (p_+ + p_- + p_0) \right\|_{op} \\ &= \|F_1(u_j) - F_1(1)\|_{op} \xrightarrow{j \rightarrow \infty} 0\end{aligned}$$

because of the continuity of F_1 in 1.

Therefore, Property 5 is also fulfilled.

Furthermore, one defines for all $a \in C^*(G)$ the function $F_a^2 : I_2 \rightarrow \mathcal{B}(L^2(K)_-)$ by

$$\begin{aligned}F_a^2(x) &:= \mathcal{F}(a)(\mathcal{P}^{-,x}) \quad \forall x \in i(0, \infty) \quad \text{and} \\ F_a^2(0) &:= \mathcal{F}(a)(\mathcal{D}_+) \circ p_+ + \mathcal{F}(a)(\mathcal{D}_-) \circ p_-.\end{aligned}$$

Using the same arguments as above, one gets the desired properties of the function F_a^2 as well.

Next, take a function $F_2 \in C_\infty(i[0, \infty), \mathcal{K}(L^2(K)_-))$ that commutes with p_+ and p_- . An operator field φ_{F_2} over \widehat{G}_{odd} meeting the Properties 1. to 5. of Theorem 3.4 has to be constructed:

Define

$$\begin{aligned}\varphi_{F_2}(\mathcal{P}^{-,x}) &:= F_2(x) \in \mathcal{B}(L^2(K)_-) \quad \forall x \in i(0, \infty), \\ \varphi_{F_2}(\mathcal{D}_+) &:= p_+ \circ F_2(0) \in \mathcal{B}(\mathcal{H}_{\mathcal{D}_+}) \quad \text{and} \\ \varphi_{F_2}(\mathcal{D}_-) &:= p_- \circ F_2(0) \in \mathcal{B}(\mathcal{H}_{\mathcal{D}_-}).\end{aligned}$$

Here again, the proof of Properties 1. to 5. is similar to the one above.

Now, take the infinite-dimensional and separable Hilbert space $\mathcal{H}_{\mathcal{D}}$ for the representations \mathcal{D}_m^+ and \mathcal{D}_m^- for $m > 1$, fixed in Chapter 4.2. Then, define for every $a \in C^*(G)$ the function $F_a^3 : I_3 \rightarrow \mathcal{B}(\mathcal{H}_{\mathcal{D}})$ by

$$\begin{aligned}F_a^3(x) &:= \mathcal{F}(a)(\mathcal{D}_x^+) \quad \forall x \in \mathbb{Z}_{>1} \quad \text{and} \\ F_a^3(x) &:= \mathcal{F}(a)(\mathcal{D}_{-x}^-) \quad \forall x \in \mathbb{Z}_{<-1}.\end{aligned}$$

Here, Property 5. of Theorem 3.4 does not emerge and the Properties 1. to 4. are obvious. Taking a function $F_3 \in C_\infty(\mathbb{Z} \setminus \{-1, 0, 1\}, \mathcal{K}(\mathcal{H}_{\mathcal{D}}))$, one has to choose

$$\begin{aligned}\varphi_{F_3}(\mathcal{D}_x^+) &:= F_3(x) \in \mathcal{B}(\mathcal{H}_{\mathcal{D}}) \quad \forall x \in \mathbb{Z}_{>1} \quad \text{and} \\ \varphi_{F_3}(\mathcal{D}_x^-) &:= F_3(-x) \in \mathcal{B}(\mathcal{H}_{\mathcal{D}}) \quad \forall x \in \mathbb{Z}_{>1}\end{aligned}$$

and it is again easy to check that φ_F complies with the properties of Theorem 3.4. □

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9 References

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